List of definitions and facts.

1. We say that p and x are **ϵ-close** when $|p - x| < \epsilon$. In other words, when the distance between p and x is less than ϵ

2. The set of all points that are ϵ-close to x is $(x - \epsilon, x + \epsilon)$.

3. A set $\mathcal{O} \subseteq \mathbb{R}$ is **open** when $\forall_{x \in \mathcal{O}} \exists_{\epsilon > 0} \, (x - \epsilon, x + \epsilon) \subseteq \mathcal{O}$.
   In other words, for any $x \in \mathcal{O}$ there is some $\epsilon > 0$ such that all points ϵ-close to x are again in $\mathcal{O}$.

4. $\mathbb{R}$ and $\emptyset$ are open (check this!). We proved in class that (a, b) is open.

5. **Any** union of open sets is always open (even infinite unions!).
   Proof: Let $\mathcal{O} = \bigcup_{i \in I} \mathcal{O}_i$ with $\mathcal{O}_i$ open. To prove: $\mathcal{O}$ is open.
   [Item 3 and WP#5 tell us that the proof should start like this:]
   Let $x \in \mathcal{O}$. To Prove: $\exists_{\epsilon > 0} \, (x - \epsilon, x + \epsilon) \subseteq \mathcal{O}$.
   $x \in \mathcal{O}$ means $x \in \mathcal{O}_i$ for some i. [Read key property of unions].
   Then $(x - \epsilon, x + \epsilon) \subseteq \mathcal{O}_i$ for some $\epsilon > 0$. [Read item 3.]
   But $\mathcal{O}_i \subseteq \mathcal{O}$ and so $(x - \epsilon, x + \epsilon) \subseteq \mathcal{O}$.
   Make sure that you can prove this if this were on a test/quiz!

6. The intersection of **finitely many** open sets is again open.
   Proof: Let $\mathcal{O} = \bigcap_{i=1}^{n} \mathcal{O}_i$ with $\mathcal{O}_i$ open, and let $x \in \mathcal{O}$. [Read key property of intersections]. Then x is also in $\mathcal{O}_i$ which is open so $\exists_{\epsilon_i} (x - \epsilon_i, x + \epsilon_i) \subseteq \mathcal{O}_i$. Now take $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$. Then $\epsilon > 0$ and $(x - \epsilon, x + \epsilon) \subseteq \mathcal{O}_i$ for every $i = 1, \ldots, n$ and thus $(x - \epsilon, x + \epsilon) \subseteq \mathcal{O}$.
   Turn in Exercise: In this proof we intersected finitely many open sets. Point out exactly which step in the proof is wrong if we intersect infinitely many open sets: $\bigcap_{i \in \mathbb{N}} \mathcal{O}_i$.

7. Let $a_1, a_2, \ldots$ be a sequence. A **tail** is a subsequence of the form $a_{K+1}, a_{K+2}, \ldots$
   So a tail is: all terms beyond some cutoff point K.

8. $a_1, a_2, \ldots$ **converges** to α when $\forall_{\epsilon > 0} \, \exists_K \, \forall_{j > K} \, |a_i - \alpha| < \epsilon$. In other words, or every $\epsilon > 0$ the sequence has a tail contained in $(\alpha - \epsilon, \alpha + \epsilon)$.
   In this case we call α the **limit** of the sequence $a_1, a_2, \ldots$.

9. α is called a **limit point** of V when (i) there is a sequence in $V - \{\alpha\}$ that converges to α. This is equivalent to (ii) $\forall_{\epsilon > 0} \, (\alpha - \epsilon, \alpha + \epsilon) \bigcap (V - \{\alpha\}) \neq \emptyset$.
   (i) $\Longrightarrow$ (ii). Let $\epsilon > 0$. Item 8 says that a tail of the sequence in (i) is in $(\alpha - \epsilon, \alpha + \epsilon)$, but the sequence is also in $V - \{\alpha\}$, so the intersection of those two sets is not empty.
   (ii) $\Longrightarrow$ (i). Construct a_n as follows. The intersection in (ii) is not empty if $\epsilon = 1/n$, so pick an element and call it a_n. Doing this for every $n \in \mathbb{N}^*$ gives a sequence $a_1, a_2, \ldots$ that meets the requirements in (i).

10. A set $V \subseteq \mathbb{R}$ is closed when

    (a) The complement of V is open.

    (b) If a sequence $a_1, a_2, \ldots$ in V converges to α then $\alpha \in V$.

    (c) V contains all of its limit points.

    (d) If $(\alpha - \epsilon, \alpha + \epsilon) \bigcap V$ is not empty for every $\epsilon > 0$ then $\alpha \in V$.

(a) $\implies$ (b) is Proposition 7.2.3.
(b) $\implies$ (a) is Proposition 7.2.4.
(b) and (c) say the same thing.
(c) and (d) are equivalent because (i) $\iff$ (ii) in the previous item.

11. Notation: $\overline{S}$ is called the **closure** of the set S

    (a) $\overline{S}$ is the union of S and all of its limit points.

    (b) $\overline{S}$ is the smallest closed set that contains S.

    (c) $\overline{S}$ is the intersection of all closed sets that contain S.

    (d) $x \in \overline{S} \iff \forall_{\epsilon>0}$ there is a point in S that is ϵ-close to x.

    (e) $x \in \overline{S} \iff \forall_{\epsilon>0}\,(x-\epsilon, x+\epsilon)$ intersects S.

    (f) $x \in \overline{S} \iff \exists$ a sequence $a_1, a_2, \ldots \in S$ that converges to x.

12. α is a limit point of S if α is in the closure of $S - \{\alpha\}$.

13. The union of *finitely many* closed sets is again closed.

14. The intersection of closed sets (even infinitely many closes sets) is closed.

    Items 13 and 14 follow immediately from: items 5, 6, 10(a) and
    De Morgan's laws: ${}^c(\bigcup S_i) = \bigcap {}^cS_i$ and ${}^c(\bigcap S_i) = \bigcup {}^cS_i$.

    Turn in exercise: Give a second proof for item 14 using only item 10(b). Let $V = \bigcap_{i \in I} V_i$ with V_i closed. Let $a_1, a_2, \ldots \in V$ and suppose that it converges to α. Then show directly that $\alpha \in V$.

15. **Turn in Exercise:** Prove that a set with one point is closed.
    Then from item 13 it follows that every finite set is closed!

16. An **interior point** of S is a point s for which $\exists_{\epsilon>0}\,(s-\epsilon, s+\epsilon) \subseteq S$.
    The definition of open in item 3 tells us that (check this!):
    (*) S is open if and only if every element of S is an interior point of S.
    Denote $\mathrm{Int}(S)$ as the set of interior points of S.
    Exercise 5 in the book asks: prove that $\mathrm{Int}(S)$ is open.

    I'll give one proof but there are many others: Let $s \in \mathrm{Int}(S)$. Then $(s-\epsilon, s+\epsilon) \subseteq S$ for some $\epsilon > 0$. But $(s-\epsilon, s+\epsilon)$ is open (item 4) so all of its elements are interior points of $(s-\epsilon, s+\epsilon)$ by (*). Then they are also interior points of S because S contains $(s-\epsilon, s+\epsilon)$.
    Hence $(s-\epsilon, s+\epsilon) \subseteq \mathrm{Int}(S)$.

    Turn in: Prove that $\mathrm{Int}(S)$ is the union of all open subsets of S
    (then we can say that $\mathrm{Int}(S)$ is the largest open subset of S).

    Hint: You need to prove that if $\mathcal{O}$ is any open subset of S and $s \in \mathcal{O}$ then $s \in \mathrm{Int}(S)$ but that is similar to the proof I just gave.

17. If S is the complement of U then $\overline{S}$ is the complement of $\mathrm{Int}(U)$.
    (Use 11(c) and De Morgan's laws).

18. Read the definition in Exercise 6. Which of (a)–(f) in item 11 would be most suitable to prove:
    The boundary of S is the intersection of $\overline{S}$ and $\overline{{}^cS}$.

19. (Ex 13 in the book). Let L be the set of limit points of S. Prove that L is closed.

    Proof: Lets prove that L is closed by using 10(c). Note: in class I tried using 10(b) but then the proof has one more step.
    To prove 10(c) for L, take a limit point α of L, then we have to prove that $\alpha \in L$, in other words: to prove that α is a limit point of S. By 9(i) that means to prove: $\exists$ sequence $b_1, b_2, \ldots$ in $S - \{\alpha\}$ that converges to α.

    If α is a limit point of L then item 9(i) says that there is a sequence $a_1, a_2, \ldots \in L - \{\alpha\}$ that converges to α. Then $\epsilon_n := |a_n - \alpha| > 0$ converges to 0. Since a_n is in L, it is a limit point of S, so there is a sequence in $S - \{a_n\}$ that converges to a_n. A tail of that sequence will be ϵ_n-close to a_n, see item 8. Take some b_n in that tail. Then $b_n \in S - \{a_n\}$ and b_n is ϵ_n-close to a_n. The distance between b_n,α is at most the distance between b_n,a_n plus the distance between a_n, α (that's called the triangle inequality). So: $|b_n - \alpha| \leq |b_n - a_n| + |a_n - \alpha| < 2\epsilon_n$. So the distance between b_n, α converges to 0, so $b_1, b_2, \ldots$ converges to α.