List of definitions and facts.

1. We say that $p$ and $x$ are $\epsilon$-close when $|p - x| < \epsilon$. In other words, when the distance between $p$ and $x$ is less than $\epsilon$.

2. The set of all points that are $\epsilon$-close to $x$ is $(x - \epsilon, x + \epsilon)$.

3. A set $O \subseteq \mathbb{R}$ is open when $\forall x \in O \exists \epsilon > 0 (x - \epsilon, x + \epsilon) \subseteq O$.
   In other words, for any $x \in O$ there is some $\epsilon > 0$ such that all points $\epsilon$-close to $x$ are again in $O$.

4. $\mathbb{R}$ and $\emptyset$ are open (check this!). We proved in class that $(a, b)$ is open.

5. Any union of open sets is always open (even infinite unions!)
   Proof: Let $O = \bigcup_{i \in I} O_i$ with $O_i$ open. To prove: $O$ is open.
   [Item 3 and WP#5 tell us that the proof should start like this:]
   Let $x \in O$. To Prove: $\exists \epsilon > 0 (x - \epsilon, x + \epsilon) \subseteq O$.
   $x \in O$ means $x \in O_i$ for some $i$. [Read key property of unions].
   Then $(x - \epsilon, x + \epsilon) \subseteq O_i$ for some $\epsilon > 0$. [Read item 3.]
   But $O_i \subseteq O$ and so $(x - \epsilon, x + \epsilon) \subseteq O$.
   Make sure that you can prove this if this were on a test/quiz!

6. The intersection of finitely many open sets is again open.
   Proof: Let $O = \bigcap_{i=1}^{n} O_i$ with $O_i$ open, and let $x \in O$. [Read key property of intersections].
   Then $x$ is also in $O_i$ which is open so $\exists \epsilon_i (x - \epsilon_i, x + \epsilon_i) \subseteq O_i$.
   Now take $\epsilon = \min(\epsilon_1, \ldots, \epsilon_n)$. Then $\epsilon > 0$ and $(x - \epsilon, x + \epsilon) \subseteq O_i$ for every $i = 1, \ldots, n$ and thus $(x - \epsilon, x + \epsilon) \subseteq O$.
   Turn in Exercise: In this proof we intersected finitely many open sets.
   Point out exactly which step in the proof is wrong if we intersect infinitely many open sets: $\bigcap_{i \in \mathbb{N}} O_i$.

7. Let $a_1, a_2, \ldots$ be a sequence. A tail is a subsequence of the form $a_{K+1}, a_{K+2}, \ldots$
   So a tail is: all terms beyond some cutoff point $K$.

8. $a_1, a_2, \ldots$ converges to $\alpha$ when $\forall \epsilon > 0 \exists K \forall j > K |a_j - \alpha| < \epsilon$. In other words, or every $\epsilon > 0$ the sequence has a tail contained in $(\alpha - \epsilon, \alpha + \epsilon)$.
   In this case we call $\alpha$ the limit of the sequence $a_1, a_2, \ldots$.

9. $\alpha$ is called a limit point of $V$ when (i) there is a sequence in $V - \{\alpha\}$ that converges to $\alpha$. This is equivalent to (ii) $\forall \epsilon > 0 (\alpha - \epsilon, \alpha + \epsilon) \cap (V - \{\alpha\}) \neq \emptyset$.
   (i) $\implies$ (ii). Let $\epsilon > 0$. Item 8 says that a tail of the sequence in (i) is in $(\alpha - \epsilon, \alpha + \epsilon)$, but the sequence is also in $V - \{\alpha\}$, so the intersection of those two sets is not empty.
   (ii) $\implies$ (i). Construct $a_n$ as follows. The intersection in (ii) is not empty if $\epsilon = 1/n$, so pick an element and call it $a_n$. Doing this for every $n \in \mathbb{N}^*$
   gives a sequence $a_1, a_2, \ldots$ that meets the requirements in (i).

10. A set $V \subseteq \mathbb{R}$ is closed when
    (a) The complement of $V$ is open.
    (b) If a sequence $a_1, a_2, \ldots$ in $V$ converges to $\alpha$ then $\alpha \in V$.
    (c) $V$ contains all of its limit points.
    (d) If $(\alpha - \epsilon, \alpha + \epsilon) \cap V$ is not empty for every $\epsilon > 0$ then $\alpha \in V$. 

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(a) \( \implies \) (b) is Proposition 7.2.3.
(b) \( \implies \) (a) is Proposition 7.2.4.
(b) and (c) say the same thing.
(c) and (d) are equivalent because (i) \( \iff \) (ii) in the previous item.

11. Notation: \( \overline{S} \) is called the closure of the set \( S \)

(a) \( \overline{S} \) is the union of \( S \) and all of its limit points.
(b) \( \overline{S} \) is the smallest closed set that contains \( S \).
(c) \( \overline{S} \) is the intersection of all closed sets that contain \( S \).
(d) \( x \in \overline{S} \iff \forall \epsilon > 0 \) there is a point in \( S \) that is \( \epsilon \)-close to \( x \).
(e) \( x \in \overline{S} \iff \forall \epsilon > 0 \) \((x - \epsilon, x + \epsilon)\) intersects \( S \).
(f) \( x \in \overline{S} \iff \exists \) a sequence \( a_1, a_2, \ldots \in S \) that converges to \( x \).

12. \( \alpha \) is a limit point of \( S \) if \( \alpha \) is in the closure of \( S \) \( - \{ \alpha \} \).

13. The union of \( \text{finitely many} \) closed sets is again closed.

14. The intersection of closed sets (even infinitely many closed sets) is closed.

Items 13 and 14 follow immediately from: items 5, 6, 10(a) and De Morgan’s laws: \( c(\bigcup S_i) = \bigcap c S_i \) and \( c(\bigcap S_i) = \bigcup c S_i \).

**Turn in exercise:** Give a second proof for item 14 using only item 10(b).
Let \( V = \bigcap_{i \in I} V_i \) with \( V_i \) closed. Let \( a_1, a_2, \ldots \in V \) and suppose that it converges to \( \alpha \). Then show directly that \( \alpha \in V \).

15. **Turn in Exercise:** Prove that a set with one point is closed.
Then from item 13 it follows that every finite set is closed!

16. An **interior point** of \( S \) is a point \( s \) for which \( \exists \epsilon > 0 \) \((s - \epsilon, s + \epsilon) \subseteq S \).

The definition of open in item 3 tells us that (check this!):

(*) \( S \) is open if and only if every element of \( S \) is an interior point of \( S \).

Denote \( \text{Int}(S) \) as the set of interior points of \( S \).

Exercise 5 in the book asks: prove that \( \text{Int}(S) \) is open.

I’ll give one proof but there are many others: Let \( s \in \text{Int}(S) \). Then \((s - \epsilon, s + \epsilon) \subseteq S \) for some \( \epsilon > 0 \). But \((s - \epsilon, s + \epsilon)\) is open (item 4) so all of its elements are interior points of \((s - \epsilon, s + \epsilon)\) by (*). Then they are also interior points of \( S \) because \( S \) contains \((s - \epsilon, s + \epsilon)\).

Hence \((s - \epsilon, s + \epsilon) \subseteq \text{Int}(S) \).

**Turn in:** Prove that \( \text{Int}(S) \) is the union of all open subsets of \( S \) (then we can say that \( \text{Int}(S) \) is the largest open subset of \( S \)).

**Hint:** You need to prove that if \( \mathcal{O} \) is any open subset of \( S \) and \( s \in \mathcal{O} \) then \( s \in \text{Int}(S) \) but that is similar to the proof I just gave.

17. If \( S \) is the complement of \( U \) then \( \overline{S} \) is the complement of \( \text{Int}(U) \).
(Use 11(c) and De Morgan’s laws).

18. Read the definition in Exercise 6. Which of (a)–(f) in item 11 would be most suitable to prove:

The boundary of \( S \) is the intersection of \( \overline{S} \) and \( \overline{\overline{S}} \).
19. (Ex 13 in the book). Let $L$ be the set of limit points of $S$. Prove that $L$ is closed.

Proof: Let’s prove that $L$ is closed by using 10(c). Note: in class I tried using 10(b) but then the proof has one more step.

To prove 10(c) for $L$, take a limit point $\alpha$ of $L$, then we have to prove that $\alpha \in L$, in other words: to prove that $\alpha$ is a limit point of $S$. By 9(i) that means to prove: $\exists$ sequence $b_1, b_2, \ldots$ in $S - \{\alpha\}$ that converges to $\alpha$.

If $\alpha$ is a limit point of $L$ then item 9(i) says that there is a sequence $a_1, a_2, \ldots \in L - \{\alpha\}$ that converges to $\alpha$. Then $\epsilon_n := |a_n - \alpha| > 0$ converges to 0. Since $a_n$ is in $L$, it is a limit point of $S$, so there is a sequence in $S - \{a_n\}$ that converges to $a_n$. A tail of that sequence will be $\epsilon_n$-close to $a_n$, see item 8. Take some $b_n$ in that tail. Then $b_n \in S - \{a_n\}$ and $b_n$ is $\epsilon_n$-close to $a_n$. The distance between $b_n, \alpha$ is at most the distance between $b_n, a_n$ plus the distance between $a_n, \alpha$ (that’s called the triangle inequality). So: $|b_n - \alpha| \leq |b_n - a_n| + |a_n - \alpha| < 2\epsilon_n$. So the distance between $b_n, \alpha$ converges to 0, so $b_1, b_2, \ldots$ converges to $\alpha$. 

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