Quiz 2, Intro Adv Math. Name:

1. Write down the definition of

(a) The **product** of sets $A$ and $B$ is $A \times B = \{(a, b) \mid a \in A, b \in B\}$

Make sure you are familiar with set notation. The notation $\{x \mid P(x)\}$
(instead of $|$ you may also use a colon) means the set of all $x$ for
which $P(x)$ is true. Spelled out, the answer above is: the set of all
pairs $(a, b)$ for which $a \in A$ and $b \in B$.

(b) A relation $R$ is called **transitive** when:

$$xRy \land yRz \implies xRz$$

(c) Let $R$ be an equivalence relation on a set $A$. The **equivalence class**
$E_x$ of an element $x \in A$ is:

$$\{y \in A \mid xRy\}$$

Again, make sure you can read and write set notation. If you spell
out this answer in words you get: “the set of all elements of $A$ that
are related to $x$ under the relation $R$”. However:

You are much better off if you **do not spell things out like that**.
It takes some effort getting used to short notation, but it will save
time during HW and tests, and makes proofs easier to read + write.

2. Let $A, B, C$ be sets and assume that

$$C - A \subseteq B \quad (1)$$

Prove that then

$$C \subseteq A \cup B. \quad (2)$$

I will give part of the proof, then you will finish the proof. First of all,
you **must know** that (2) is equivalent to (for all $x$):

$$x \in C \implies x \in A \cup B \quad (3)$$

Lets prove (3) with WP#4 which tells us to do this:
Assume $x \in C$ and $\neg (x \in A \cup B)$. \ (*)
To prove: a contradiction.

L1: \ (*) is equivalent to $x \in C$ and $x \notin A \land x \notin B$ \ (to see this, look at
De Morgan’s laws)
So $x \in C$ and $x \notin A$, so $x \in C - A$ and then $x \in B$ by (1) but that
contradicts the last item $x \notin B$ on L1.
Writing Proofs.

1. **Direct proof for** \( p \implies q \).
   - Assume: \( p \). To prove: \( q \).

2. **Proving** \( p \implies q \) **by contrapositive**.
   - Assume: \( \neg q \). To prove: \( \neg p \).

3. **Proving** \( S \) **by contradiction**.
   - Assume: \( \neg S \). To prove: a contradiction.

4. **Proving** \( p \implies q \) **by contradiction**.
   - Assume: \( p \) and \( \neg q \). To prove: a contradiction.

5. **Direct proof for a** \( \forall x \in A P(x) \) **statement**.
   - To ensure you prove \( P(x) \) for all (rather than for some) \( x \) in \( A \), do this:
     - **Start your proof with:** Let \( x \in A \). To prove: \( P(x) \).

6. **Direct proof for** \( \exists x \in A P(x) \) **statement**.
   - Take \( x := \) [write down an expression that is in \( A \), and satisfies \( P(x) \)].

7. **Proving** \( \forall x \in A P(x) \) **by contradiction**.
   - Assume: \( x \in A \) and \( \neg P(x) \). To prove: a contradiction.

8. **Proving** \( \exists x \in A P(x) \) **by contradiction**.
   - Assume: \( \neg P(x) \) for every \( x \in A \). To prove: a contradiction.

9. **Proving** \( S \) **by cases**.
   - Suppose for example a statement \( p \) can help to prove \( S \). Write two proofs:
     - Case 1: Assume \( p \). To prove: \( S \).
     - Case 2: Assume \( \neg p \). To prove \( S \).

10. **Proving** \( p \land q \)
    - Write two separate proofs: To prove: \( p \). To prove: \( q \).

11. **Proving** \( p \iff q \)
    - Write two proofs. To prove: \( p \implies q \) To prove: \( q \implies p \).

12. **Proving** \( p \lor q \)
    - Method (1): Assume \( \neg p \). To prove: \( q \).
    - Method (2): Assume \( \neg q \). To prove: \( p \).
    - Method (3): Assume \( \neg p \) and \( \neg q \). To prove: a contradiction.

13. **Using** \( p \lor q \) **to prove another statement** \( r \).
    - Write two proofs:
      - Assume \( p \). To prove \( r \).
      - Assume \( q \). To prove \( r \).

14. **How to use a for-all statement** \( \forall x \in A P(x) \).
    - You need to produce an element of \( A \), then use \( P \) for that element.

15. If you want to **use an exists statement** like \( \exists x \in A P(x) \) to prove another statement, then you **may not choose** \( x \). All you know is \( x \in A \) and \( P(x) \).