Answers to sample questions, Intro Advanced Math

- Suppose that A is a set and that for every set B we have $A \subseteq B$. Then prove that $A = \emptyset$.

  Lets follow the idea in WP#14 and choose $B = \emptyset$. Since $A \subseteq B$ is true for every B, it is also true for our choice. So $A \subseteq \emptyset$ and we showed in a previous exercise that then $A = \emptyset$.

- Consider the statement S:

  $$S: \quad \forall_{p\in\mathbb{R}} \ \ p > 0 \implies 10^{11} \times p > 1$$

  Write down $\neg S$, the negation of statement S.

  $$\neg S: \quad \exists_{p\in\mathbb{R}} \ \ p > 0 \wedge \ 10^{11} \times p \leq 1$$

  For which of the following statements can you give a proof, S or $\neg S$?
  WP#6 says that we can prove an exists-statement by writing down an example. So here is a 1-line proof of $\neg S$:

  Take $p := 10^{-20}$. Then $p > 0$ and $10^{11} \times p \leq 1$.

  (Don't worry if your example is different. As long as you: ***gave an example***, your example is > 0, and 10^{11} times your example is ≤ 1, then your proof is correct.)

- Prove that $n^2 + n$ is even for any integer n.

  There are a numerous valid proofs. Here is a proof by cases:
  Case n is even. Then $n^2 + n$ is even+even which is even.
  Case n is odd. Then $n^2 + n$ is odd+odd which is even.

- If n is odd then prove that $n^2 - 1$ is divisible by 8.

  If n is odd then (note: these are the types of things you need to memorize!) we can write $n = 2k + 1$ for some integer k.
  Then $n^2 - 1 = (2k)^2 + 2(2k) + 1 - 1 = 4(k^2 + k)$. But by $k^2 + k$ is even by the previous exercise, so $n^2 - 1 = 4(k^2 + k)$ must be divisible by $4 \cdot 2 = 8$.

- Let $S(n) = 1 + 2 + 2^2 + \cdots + 2^n = \sum_{k=0}^{n} 2^k$. Prove that $S(n) = 2^{n+1} - 1$.

  Note: there are several ways to prove this, but the intended proof is to use induction.

  Task one: Prove the formula for $n = 1$. In this case $S(n) = 1 + 2^1$ while the right-hand-side is $2^2 - 1$ which is the same number.

  Task two: Prove the following: $S(n) = 2^{n+1} - 1 \Longrightarrow S(n+1) = 2^{n+1+1} - 1$.
  Assume $S(n) = 2^{n+1} - 1$ (this assumption is the "induction hypothesis").
  To Prove: $S(n + 1) = 2^{n+1+1} - 1$.
  The left-hand-side is $S(n+1) = 1 + 2 + 2^2 + \cdots + 2^n + 2^{n+1} = S(n) + 2^{n+1}$ which is $2^{n+1} - 1 + 2^{n+1}$ by the induction hypothesis.
  The right-hand-side $2^{n+1+1} - 1 = 2^{n+1} + 2^{n+1} - 1$ equals the left-hand-side.