Test 2 sample questions, with answers

1. Let R be a relation on a set S. Give the definition: R is a *partial ordering* when: Answer:

   (a) $\forall_{x \in S} \ \ xRx$ [Reflexive]

   (b) $\forall_{x,y \in S} \ \ xRy \wedge yRx \implies x = y$ [Anti-symmetric]

   (c) $\forall_{x,y,z \in S} \ \ xRy \wedge yRz \implies xRz$ [Transitive]

2. If R is a partial ordering, then R is a total ordering if it has one more property, namely:

   $\forall_{x,y \in S} \ \ x \neq y \implies xRy \vee yRx$

3. Line 1. Suppose that R is a partial ordering, but not a total ordering.
   Line 2. To prove: there exists a non-empty subset $A \subseteq S$ for which A does not have a minimal element.

   Given in Line 1: items (a),(b),(c) in Exercise 1 and the negation of the statement in Exercise 2 which is: (d) $\exists_{x,y \in S} \ \ x \neq y \wedge (\neg\, xRy \wedge \neg\, yRx)$.

   [T.P.: There is a set $A \subseteq S$ with certain properties. The Writing Proofs handout tells us that we have to write: Take $A := \ldots$. This A needs to be a subset of S so what we fill in on those dots must be: {some element(s) of S}. Did we encounter any? Well, yes, (d) says that there are x, y in S with some properties. This gives the idea for the next line:]

   Take $A := \{x, y\}$ with x, y as in statement (d).
   Remains to prove: A does not have a minimal element. Proof: x is not minimal because $\neg\, xRy$ and y is not minimal because $\neg\, yRx$.

4. Give the definition of an equivalence relation:

   A relation that is: Reflexive (Ex 1(a)), Symmetric ($xRy \implies yRx$), and Transitive (Ex 1(c)).

5. Let $f : A \to B$ be a function. We now define a relation R on A as follows: xRy is true if and only if $f(x) = f(y)$. Is this relation:

   (a) Reflexive? Yes. xRx means $f(x) = f(x)$ which is always true.

   (b) Symmetric? Yes. xRy means $f(x) = f(y)$ which implies $f(y) = f(x)$ which is the same as yRx.

   (c) Transitive? Yes. xRy and yRz means $f(x) = f(y)$ and $f(y) = f(z)$ but then $f(x) = f(z)$ so xRz.

   (d) An equivalence relation? Yes.

   (e) If R is a partial ordering then prove that f is injective.

   Given: R is a partial ordering, i.e. R is reflexive, anti-symmetric, and transitive. Note that we already proved that R is reflexive and transitive, so the only new information we get here is that R is anti-symmetric, i.e. $xRy \wedge yRx \implies x = y$ (1).

To prove: f is injective, i.e. $f(x) = f(y) \implies x = y$.
Assume $f(x) = f(y)$. T.P. $x = y$.
Proof: $f(x) = f(y)$ means xRy but R is symmetric so yRx.
Then $x = y$ by (1).

6. Suppose that $f : A \to B$ is not injective. Show that $\text{card}(A) \geq 2$.

   [Know the definition of injective! Know how to compute negations!]

   f is not injective means: $\exists_{a_1,a_2 \in A}\ f(a_1) = f(a_2) \wedge a_1 \neq a_2$.
   Take such a_1, a_2. Then $a_1, a_2 \in A$ and $a_1 \neq a_2$ so A has at least 2 elements.

7. Give a function $f : \mathbb{N}^* \to \mathbb{N}^*$ that is injective but not surjective.

   There are many correct answers. The first story in the hotel infinity handout gives this function $f(n) = n + 1$.

   Give a function $g : \mathbb{N}^* \to \mathbb{N}^*$ that is surjective but not injective.

   Attempt #1. Lets do the opposite of f and take $g(n) := n - 1$. This does not work because then $g(1) \notin \mathbb{N}^*$.
   Attempt #2. Lets modify this and take $g(n) = 1$ if $n = 1$ and $g(n) = n-1$ if $n > 1$. This g is surjective but not injective.

8. If A is any countably infinite set (item 5 in the handout) then show that there exists a function from A to A that is injective but not surjective.

   A is countably infinite when there exists a bijection $f : \mathbb{N}^* \to A$. Then $A = f(\mathbb{N}^*) = \{f(1), f(2), f(3), \ldots\}$. We can now do the same as in Hotel Infinity, namely, we can take a function $A \to A$ that sends $f(n)$ to $f(n+1)$.

9. If S is any infinite set, then use items 17 and 5 from the handout to show that there exists a function from S to S that is injective but not surjective.

   If S is infinite then S has a countably infinite subset $A \subseteq S$. As in the previous question, we can write $A = \{f(1), f(2), f(3), \ldots\}$. Now make a function $h : S \to S$ as follows. Let $s \in S$. If $s \in A$ then $s = f(n)$ for some n, then we define $h(s) = f(n + 1)$. If $s \notin A$ then we define $h(s) = s$.

Lets end this handout with a puzzle: Suppose that

(a) S and T are subsets of $\mathbb{R}$

and

(b) For every $s \in S$ and every $t \in T$ we have $s > t^2$.

Can we conclude from this:

(c) Every element of S is positive?

If yes, then prove (a)$\wedge$(b) $\implies$ (c).
If not, explain why.