Test 2 sample questions, with answers

1. Let $R$ be a relation on a set $S$. Give the definition: $R$ is a partial ordering when: Answer:

(a) $\forall x \in S \ x Rx$ [Reflexive]
(b) $\forall x, y \in S \ x Ry \land y Rx \implies x = y$ [Anti-symmetric]
(c) $\forall x, y, z \in S \ x Ry \land y Rz \implies x Rz$ [Transitive]

2. If $R$ is a partial ordering, then $R$ is a total ordering if it has one more property, namely:

$\forall x, y \in S \ x \neq y \implies x Ry \lor y Rx$

3. Line 1. Suppose that $R$ is a partial ordering, but not a total ordering.
Line 2. To prove: there exists a non-empty subset $A \subseteq S$ for which $A$ does not have a minimal element.

Given in Line 1: items (a),(b),(c) in Exercise 1 and the negation of the statement in Exercise 2 which is: (d) $\exists x, y \in S \ x \neq y \land (\neg x Ry \land \neg y Rx)$.

[T.P.: There is a set $A \subseteq S$ with certain properties. The Writing Proofs handout tells us that we have to write: Take $A := \ldots$ This $A$ needs to be a subset of $S$ so what we fill in on those dots must be: {some element(s) of $S$}. Did we encounter any? Well, yes, (d) says that there are $x, y$ in $S$ with some properties. This gives the idea for the next line:]

Take $A := \{x, y\}$ with $x, y$ as in statement (d).
Remains to prove: $A$ does not have a minimal element. Proof: $x$ is not minimal because $\neg x Ry$ and $y$ is not minimal because $\neg y Rx$.

4. Give the definition of an equivalence relation:

A relation that is: Reflexive (Ex 1(a)), Symmetric ($x Ry \implies y Rx$), and Transitive (Ex 1(c)).

5. Let $f : A \rightarrow B$ be a function. We now define a relation $R$ on $A$ as follows: $x Ry$ is true if and only if $f(x) = f(y)$. Is this relation:

(a) Reflexive? Yes. $x Rx$ means $f(x) = f(x)$ which is always true.
(b) Symmetric? Yes. $x Ry$ means $f(x) = f(y)$ which implies $f(y) = f(x)$ which is the same as $y Rx$.
(c) Transitive? Yes. $x Ry$ and $y Rz$ means $f(x) = f(y)$ and $f(y) = f(z)$ but then $f(x) = f(z)$ so $x Rz$.
(d) An equivalence relation? Yes.
(e) If $R$ is a partial ordering then prove that $f$ is injective.

Given: $R$ is a partial ordering, i.e. $R$ is reflexive, anti-symmetric, and transitive. Note that we already proved that $R$ is reflexive and transitive, so the only new information we get here is that $R$ is antisymmetric, i.e. $x Ry \land y Rx \implies x = y$ (1).
To prove: \( f \) is injective, i.e. \( f(x) = f(y) \implies x = y \).

Assume \( f(x) = f(y) \). T.P. \( x = y \).

Proof: \( f(x) = f(y) \) means \( xRy \) but \( R \) is symmetric so \( yRx \).

Then \( x = y \) by (1).

6. Suppose that \( f : A \to B \) is not injective. Show that \( \text{card}(A) \geq 2 \).

[Know the definition of injective! Know how to compute negations!]

\( f \) is not injective means: \( \exists a_1, a_2 \in A \) \( f(a_1) = f(a_2) \land a_1 \neq a_2 \).

Take such \( a_1, a_2 \). Then \( a_1, a_2 \in A \) and \( a_1 \neq a_2 \) so \( A \) has at least 2 elements.

7. Give a function \( f : \mathbb{N}^* \to \mathbb{N}^* \) that is injective but not surjective.

There are many correct answers. The first story in the hotel infinity handout gives this function \( f(n) = n + 1 \).

Give a function \( g : \mathbb{N}^* \to \mathbb{N}^* \) that is surjective but not injective.

Attempt #1. Let's do the opposite of \( f \) and take \( g(n) := n - 1 \). This does not work because then \( g(1) \notin \mathbb{N}^* \).

Attempt #2. Let's modify this and take \( g(n) = 1 \) if \( n = 1 \) and \( g(n) = n - 1 \) if \( n > 1 \). This \( g \) is surjective but not injective.

8. If \( A \) is any countably infinite set (item 5 in the handout) then show that there exists a function from \( A \) to \( A \) that is injective but not surjective.

\( A \) is countably infinite when there exists a bijection \( f : \mathbb{N}^* \to A \). Then \( A = f(\mathbb{N}^*) = \{ f(1), f(2), f(3), \ldots \} \). We can now do the same as in Hotel Infinity, namely, we can take a function \( A \to A \) that sends \( f(n) \) to \( f(n+1) \).

9. If \( S \) is any infinite set, then use items 17 and 5 from the handout to show that there exists a function from \( S \) to \( S \) that is injective but not surjective.

If \( S \) is infinite then \( S \) has a countably infinite subset \( A \subseteq S \). As in the previous question, we can write \( A = \{ f(1), f(2), f(3), \ldots \} \). Now make a function \( h : S \to S \) as follows. Let \( s \in S \). If \( s \in A \) then \( s = f(n) \) for some \( n \), then we define \( h(s) = f(n+1) \). If \( s \notin A \) then we define \( h(s) = s \).

Let's end this handout with a puzzle: Suppose that

(a) \( S \) and \( T \) are subsets of \( \mathbb{R} \)

and

(b) For every \( s \in S \) and every \( t \in T \) we have \( s > t^2 \).

Can we conclude from this:

(c) Every element of \( S \) is positive?

If yes, then prove (a) \( \land \) (b) \( \implies \) (c).

If not, explain why.