Intro Advanced Math, definitions to memorize for test 1

You need to make sure to know all these definitions for Friday's test.

There will be questions like "write the definition of ..." To get credit for such questions, you must give a precise **formal** definition (do not write an intuitive interpretation). You will need precise formal definitions for the proof-questions as well (see examples below).

1. Sets are the only objects in mathematics that are not defined in terms of previously-defined objects. Instead, they are described indirectly by stating their properties as we will see later in Chapter 5. Given any set A and any object x, the phrase "$x \in A$" is a statement (which means: it is either True or False). It is pronounced as: x is an element of A.
   The negation of this statement is $x \notin A$ (x is not an element of A).

2. **Set equality** is defined by the following rule:

   $$A = B \quad \text{means} \quad x \in A \Longleftrightarrow x \in B. \tag{1}$$

   Strictly speaking it is $\forall_x \, (x \in A \Longleftrightarrow x \in B)$ but this $\forall_x$ is often omitted.

3. **Subset:**

   $$A \subseteq B \quad \text{means} \quad x \in A \Longrightarrow x \in B. \tag{2}$$

   Strictly speaking it is $\forall_x \, (x \in A \Longrightarrow x \in B)$ but we again omitted $\forall_x$.
   The notation $\subset$ means the same as $\subseteq$.

4. Since $p \Longleftrightarrow q$ is equivalent to $(p \Longrightarrow q) \wedge (q \Longrightarrow p)$ it follows from (1)+(2) that

   $$A = B \quad \text{is equivalent to} \quad A \subseteq B \ \wedge \ B \subseteq A \tag{3}$$

   So to prove sets are equal you can use either (1) or (3).

5. The **empty set** is denoted as $\emptyset$ and is defined by $\forall_x \; x \notin \emptyset$. Then (1) says

   $$A = \emptyset \quad \text{means} \quad \forall_x \; x \notin A. \tag{4}$$

   One way to prove $A = \emptyset$ starts with:
   Let x be any object. To prove $x \notin A$.

   A proof by contradiction (WP#3) for $A = \emptyset$ starts with:
   Assume $x \in A$. To prove: a contradiction.

6. Example: Suppose $A \subseteq \emptyset$. Then prove: $A = \emptyset$.
   Proof: Let $x \in A$. To prove: a contradiction.
   From $x \in A$ and $A \subseteq \emptyset$ it follows[1] that $x \in \emptyset$ which is always false.

[1] To see this you need to know (2)!

7. Example: Let A be a set. Then prove $\emptyset \subseteq A$.
To Prove: $x \in \emptyset \Longrightarrow x \in A$.
Proof: $x \in \emptyset$ is always false, and "false$\Longrightarrow$anything" is always true.
[We used (2) in the T.P. statement, and (4) + truth tables for the proof.]
[Text in square brackets [ ] are additional comments that you don't need to write in your proofs.]

8. Definition of the **power set** (the set of all subsets): $P(A) = \{S \ : \ S \subseteq A\}$ or equivalently:
$$S \subseteq A \iff S \in P(A) \tag{5}$$

9. Example: $\emptyset \subseteq A$ and hence $\emptyset \in P(A)$. Likewise $A \subseteq A$ and so $A \in P(A)$. In particular $P(A)$ is never empty.
[As discussed in class, $P(\emptyset) = \{\emptyset\}$ is not equal to $\emptyset$. Analogy: "a bag with an empty bag in it" $\neq$ "an empty bag".]

10. Example: Suppose $P(A) \subseteq P(B)$. Prove $A \subseteq B$.
Proof: $A \subseteq A$ so $A \in P(A)$ but $P(A) \subseteq P(B)$ so $A \in P(B)$ hence $A \subseteq B$.
[First we used (5), then used (2) for $P(A) \subseteq P(B)$, then used (5) again. To write proofs, you really need to know the definitions (1),(2),...!]

11. (a, b) is an **ordered pair**. For example $\{1, 2\} = \{2, 1\}$ but $(1, 2) \neq (2, 1)$.

12. $A \times B = \{(a, b) \ : \ a \in A, b \in B\}$ = the set of all pairs (a, b) for all $a \in A$ and all $b \in B$. So
$$(a, b) \in A \times B \quad \text{is the same as} \quad a \in A \wedge b \in B \tag{6}$$

13. Exercise: Show that $A \times \emptyset = \emptyset$. Proof:
An element of $A \times \emptyset$ is a pair: (an element of A, an element of $\emptyset$). There are no such pairs because $\emptyset$ has no elements.

14. Definition of $\bigcap$ $\bigcup$ and $\setminus$ (instead of $A \setminus B$ you may also write $A - B$)
$$x \in A \bigcap B \iff (x \in A \ \wedge \ x \in B) \tag{7}$$
$$x \in A \bigcup B \iff (x \in A \ \vee \ x \in B) \tag{8}$$
$$x \in \bigcap_{i \in I} A_i \iff \forall_{i \in I} \ x \in A_i \tag{9}$$
$$x \in \bigcup_{i \in I} A_i \iff \exists_{i \in I} \ x \in A_i \tag{10}$$
$$x \in A \setminus B \iff (x \in A \ \wedge \ x \notin B) \tag{11}$$

15. Example: Prove $(A \setminus B) \bigcap B = \emptyset$.
[Read item 5] Proof: Assume $x \in (A \setminus B) \bigcap B$. To prove: a contradiction. By (7) we get $x \in A \setminus B$ and $x \in B$. Then by (11) we get $x \in A$ and $x \notin B$ which contradicts $x \in B$.

16. Make sure to know truth tables, and how to negate any statement.