Test 3, Nov 18 2019.

1. For each of the following subsets of $\mathbb{R}$, mention if it is *open, closed, both*, or *neither*. For each set A that is not closed, write down its closure $\overline{A}$ (no proofs necessary).

   $\emptyset$

   $\mathbb{R}$

   $(0, \infty)$

   $(x-\epsilon, x) \bigcup (x, x+\epsilon) \quad$ (where $\epsilon > 0$)

   $\mathbb{Q}$

   For the next questions, please don't write your answers on this page, there isn't enough room.

2. Let $S \subseteq \mathbb{R}$ and let P be the statement $\exists_{x \in S} \forall_{\epsilon > 0} \; (x-\epsilon, x+\epsilon) \not\subseteq S$.
   Spell out $\neg P$ (the negation of P).
   In words, *as brief as possible* (look at attached list of facts if necessary), what does $\neg P$ say about S?

3. If $S \subseteq T \subseteq \mathbb{R}$ then show that $\text{Int}(S) \subseteq \text{Int}(T)$.

4. Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Suppose that S is dense.
   Show that $S - \{x\}$ is also dense.

   Hint: There are long proofs but there is also a short proof. Look carefully at the list of facts until you found the very best item to apply to this question.

5. Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Suppose that $S \bigcup \{x\}$ is open. Show that $x \in \overline{S}$.

If you run out of time then leave Ex 5 as a take home question.

For any question, if you can't find a proof, you should still write down the first lines in the proof that you might find from the Writing Proofs guidelines.

List of definitions and facts. Writing clear proofs means you should cite each item number(letter) that you use, precisely at the step where you use it.

1. We say that p and x are **ϵ-close** when $|p-x|<\epsilon$.
2. The set of all points that are ϵ-close to x is $(x-\epsilon, x+\epsilon)$.
3. A set $\mathcal{O} \subseteq \mathbb{R}$ is **open** when $\forall_{x\in\mathcal{O}} \exists_{\epsilon>0}\, (x-\epsilon, x+\epsilon) \subseteq \mathcal{O}$.
4. $\mathbb{R}$ and $\emptyset$ and any interval of the form (a,b) is open.
5. **Any** union of open sets is always open (even infinite unions!).
6. The intersection of **finitely many** open sets is again open.
7. Let $a_1, a_2, \ldots$ be a sequence. A **tail** is a subsequence of the form $a_{K+1}, a_{K+2}, \ldots$ So a tail is: all terms beyond some cutoff point K.
8. $a_1, a_2, \ldots$ **converges** to α when $\forall_{\epsilon>0}\, \exists_K\, \forall_{j>K}\; |a_i-\alpha|<\epsilon$. In other words, or every $\epsilon>0$ the sequence has a tail contained in $(\alpha-\epsilon, \alpha+\epsilon)$.
   In this case we call α the **limit** of the sequence $a_1, a_2, \ldots$.
9. α is called a **limit point** of V when (i) there is a sequence in $V-\{\alpha\}$ that converges to α. This is equivalent to (ii) $\forall_{\epsilon>0}\, (\alpha-\epsilon, \alpha+\epsilon) \bigcap (V-\{\alpha\}) \neq \emptyset$.
10. A set $V \subseteq \mathbb{R}$ is **closed** when
    (a) The complement of V is open.
    (b) If a sequence $a_1, a_2, \ldots$ in V converges to α then $\alpha \in V$.
    (c) V contains all of its limit points.
    (d) If $(\alpha-\epsilon, \alpha+\epsilon) \bigcap V$ is not empty for every $\epsilon>0$ then $\alpha \in V$.
11. Notation: $\overline{S}$ is called the **closure** of the set S
    (a) $\overline{S}$ is the union of S and all of its limit points.
    (b) $\overline{S}$ is the smallest closed set that contains S.
    (c) $\overline{S}$ is the intersection of all closed sets that contain S.
    (d) $x \in \overline{S} \iff \forall_{\epsilon>0}$ there is a point in S that is ϵ-close to x.
    (e) $x \in \overline{S} \iff \forall_{\epsilon>0}\, (x-\epsilon, x+\epsilon)$ intersects S.
    (f) $x \in \overline{S} \iff \exists$ a sequence $a_1, a_2, \ldots \in S$ that converges to x.
12. α is a limit point of S if α is in the closure of $S-\{\alpha\}$.
13. The union of **finitely many** closed sets is again closed.
14. **Any** intersection of closed sets (even infinitely many) is closed.
15. If $S \subseteq \mathbb{R}$ has finitely many elements then S is closed.

16. An **interior point** of S is a point s for which $\exists_{\epsilon>0}\,(s-\epsilon, s+\epsilon) \subseteq S$. Denote $\text{Int}(S)$ as the set of interior points of S.

    (a) S is open $\iff$ every element of S is an interior point of S.

    (b) $\text{Int}(S)$ is open.

    (c) $\text{Int}(S)$ is the union of all open subsets of S.

    (d) If S is the complement of U then $\overline{S}$ is the complement of $\text{Int}(U)$.

17. A point $s \in \mathbb{R}$ is a **boundary point** of S if

$$\forall_{\epsilon>0} \left((s-\epsilon, s+\epsilon) \bigcap S \neq \emptyset \ \text{ and } \ (s-\epsilon, s+\epsilon) \bigcap {}^{c}S \neq \emptyset \right)$$

    The boundary of S is the set of all boundary points of S. It also equals the intersection of $\overline{S}$ and $\overline{{}^{c}S}$.

18. (Ex 13 in the book). The set of limit points of any set S is closed.

19. A subset $S \subseteq \mathbb{R}$ is called **dense** if (definition is also in Ex 12 in the book) for every $x \in \mathbb{R}$ and every $\epsilon > 0$ the interval $(x-\epsilon, x+\epsilon)$ contains an element of S.

    (a) S dense $\iff \overline{S} = \mathbb{R}$

    (b) S dense $\iff$ For every $\alpha \in \mathbb{R}$ there exists a sequence $a_1, a_2, \ldots \in S$ that converges to α.

    (c) S dense $\iff$ Every non-empty open set has at least one element of S.

    (d) S dense $\iff$ Every non-empty open set contains infinitely many elements of S.

    (e) $\mathbb{Q}$ is dense

    (f) If A is countable then $\mathbb{R} - A$ is dense.

20. S is a **discrete** set if $\forall_{s\in S}\exists_{\epsilon>0}\ (s-\epsilon, s+\epsilon) \bigcap S = \{s\}$.

21. Every finite set is discrete.

22. Every discrete set is countable.

Writing Proofs.

1. **Direct proof for $p \implies q$.**
   Assume: p. To prove: q.

2. **Proving $p \implies q$ by contrapositive**.
   Assume: $\neg q$. To prove: $\neg p$.

3. **Proving S by contradiction**.
   Assume: $\neg S$. To prove: a contradiction.

4. **Proving $p \implies q$ by contradiction**.
   Assume: p and $\neg q$. To prove: a contradiction.

5. **Direct proof for a $\forall_{x \in A} P(x)$ statement**.
   To ensure you prove $P(x)$ for *all* (rather than for *some*) x in A, do this:
   Start your proof with: Let $x \in A$. To prove: $P(x)$.

6. **Direct proof for $\exists_{x \in A} P(x)$ statement**.
   Take $x :=$ [write down an expression that is in A, and satisfies $P(x)$].

7. **Proving $\forall_{x \in A} P(x)$ by contradiction.**
   Assume: $x \in A$ and $\neg P(x)$. To prove: a contradiction.

8. **Proving $\exists_{x \in A} P(x)$ by contradiction.**
   Assume: $\neg P(x)$ for every $x \in A$. To prove: a contradiction.

9. **Proving S by cases**.
   Suppose for example a statement p can help to prove S. Write two proofs:
   Case 1: Assume p. To prove: S.
   Case 2: Assume $\neg p$. To prove S.

10. **Proving $p \wedge q$**
    Write two separate proofs: To prove: p. To prove: q.

11. **Proving $p \iff q$**
    Write two proofs. To prove: $p \implies q$ To prove: $q \implies p$.

12. **Proving $p \vee q$**
    Method (1): Assume $\neg p$. To prove: q.
    Method (2): Assume $\neg q$. To prove: p.
    Method (3): Assume $\neg p$ and $\neg q$. To prove: a contradiction.

13. **Using $p \vee q$ to prove another statement r**.
    Write two proofs:
    Assume p. To prove r.
    Assume q. To prove r.

14. **How to use a for-all statement $\forall_{x \in A} P(x)$**.
    You need to produce an element of A, then use P for that element.

15. If you want to **use an exists statement** like $\exists_{x \in A} P(x)$ to prove another statement, then you *may not choose* x. All you know is $x \in A$ and $P(x)$.