Test 3, Nov 18 2019, Answers

1. For each of the following subsets of $\mathbb{R}$, mention if it is open, closed, both, or neither. If it is not closed, write down its closure.

   $\emptyset$ both open and closed.

   $\mathbb{R}$ both open and closed.

   $(0, \infty)$ open. Closure $= [0, \infty)$.

   $(x-\epsilon, x) \bigcup (x, x+\epsilon)$ open. Closure $= [x-\epsilon, x+\epsilon]$.

   $\mathbb{Q}$ neither open nor closed. Closure $= \mathbb{R}$.

2. Let $S \subseteq \mathbb{R}$ and let P be the statement $\exists_{x \in S} \forall_{\epsilon > 0}\ (x-\epsilon, x+\epsilon) \not\subseteq S$. Spell out $\neg P$ (the negation of P). What does $\neg P$ say about S?

   The negation of P says $\forall_{x \in S} \exists_{\epsilon > 0}\ (x-\epsilon, x+\epsilon) \subseteq S$.
   In words: S is open. (See item 3, for the final you must memorize basic definitions like that).

3. If $S \subseteq T \subseteq \mathbb{R}$ then show that $\text{Int}(S) \subseteq \text{Int}(T)$.

   [You **must know** that proving $\subseteq$ can be done by starting like this:]
   Let $s \in \text{Int}(S)$. To prove $s \in \text{Int}(T)$.
   The definition in item 16 says $\exists_{\epsilon > 0}\ (s-\epsilon, s+\epsilon) \subseteq S$. But $S \subseteq T$ so $\exists_{\epsilon > 0}\ (s-\epsilon, s+\epsilon) \subseteq S \subseteq T$. Then $s \in \text{Int}(T)$ (again item 16).

   Proof #2: $\text{Int}(T)$ is the union of all open subsets of T, one of which is $\text{Int}(S) \subseteq S \subseteq T$.

4. Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Suppose that S is dense.
   Show that $S - \{x\}$ is also dense.

   Item 19(d) says that every non-empty open set contains infinitely many elements from S. If you delete x then there are still infinitely many left. So every non-empty open set contains infinitely many elements of $S - \{x\}$. Then $S - \{x\}$ is dense by 19(d).

5. Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Suppose that $S \bigcup \{x\}$ is open. Show that $x \in \overline{S}$.

   From item 3 we get $(x-\epsilon, x+\epsilon) \subseteq S \bigcup \{x\}$ for some $\epsilon > 0$.
   Then $(x-\epsilon, x) \bigcup (x, x+\epsilon)$ is a subset of S.
   But recall from Ex 1 that x is a limit point of $(x-\epsilon, x) \bigcup (x, x+\epsilon)$.
   Then x is also a limit point of S. Then use item 11(a).

Good news: the quiz to add points to test 2 went very well.
Bad news: test 3 did not go well.
Good news: there is enough time to have a similar quiz for test 3.