Test 3, Nov 18 2019, Answers

1. For each of the following subsets of $\mathbb{R}$, mention if it is open, closed, both, or neither. If it is not closed, write down its closure.

- $\emptyset$: both open and closed.
- $\mathbb{R}$: both open and closed.
- $(0, \infty)$: open. Closure = $[0, \infty)$. 
- $(x - \epsilon, x + \epsilon) \cup (x, x + \epsilon)$: open. Closure = $[x - \epsilon, x + \epsilon]$. 
- $\mathbb{Q}$: neither open nor closed. Closure = $\mathbb{R}$.

2. Let $S \subseteq \mathbb{R}$ and let $P$ be the statement $\exists_{x \in S} \forall_{\epsilon > 0} (x - \epsilon, x + \epsilon) \nsubseteq S$.

Spell out $\neg P$ (the negation of $P$). What does $\neg P$ say about $S$?

The negation of $P$ says $\forall_{x \in S} \exists_{\epsilon > 0} (x - \epsilon, x + \epsilon) \subseteq S$.

In words: $S$ is open. (See item 3, for the final you must memorize basic definitions like that).

3. If $S \subseteq T \subseteq \mathbb{R}$ then show that $\text{Int}(S) \subseteq \text{Int}(T)$.

[You must know that proving $\subseteq$ can be done by starting like this:] 
Let $s \in \text{Int}(S)$. To prove $s \in \text{Int}(T)$.
The definition in item 16 says $\exists_{\epsilon > 0} (s - \epsilon, s + \epsilon) \subseteq S$. But $S \subseteq T$ so $\exists_{\epsilon > 0} (s - \epsilon, s + \epsilon) \subseteq S \subseteq T$. Then $s \in \text{Int}(T)$ (again item 16).

Proof #2: $\text{Int}(T)$ is the union of all open subsets of $T$, one of which is $\text{Int}(S) \subseteq S \subseteq T$.

4. Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Suppose that $S$ is dense.
Show that $S \setminus \{x\}$ is also dense.
Item 19(d) says that every non-empty open set contains infinitely many elements from $S$. If you delete $x$ then there are still infinitely many left. So every non-empty open set contains infinitely many elements of $S \setminus \{x\}$. Then $S \setminus \{x\}$ is dense by 19(d).

5. Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Suppose that $S \cup \{x\}$ is open. Show that $x \in \overline{S}$.

From item 3 we get $(x - \epsilon, x + \epsilon) \subseteq S \cup \{x\}$ for some $\epsilon > 0$.
Then $(x - \epsilon, x) \cup (x, x + \epsilon)$ is a subset of $S$.
But recall from Ex 1 that $x$ is a limit point of $(x - \epsilon, x) \cup (x, x + \epsilon)$. Then $x$ is also a limit point of $S$. Then use item 11(a).

Good news: the quiz to add points to test 2 went very well.
Bad news: test 3 did not go well.
Good news: there is enough time to have a similar quiz for test 3.