Test 3, Nov 18 2019, More answers to Ex 3–5

3. If $S \subseteq T \subseteq \mathbb{R}$ then show that $\text{Int}(S) \subseteq \text{Int}(T)$.

   Proof #1: Let $s \in \text{Int}(S)$. To prove $s \in \text{Int}(T)$.
   The definition in item 16 says $\exists_{\epsilon>0}\ (s-\epsilon, s+\epsilon) \subseteq S$. But $S \subseteq T$ so $\exists_{\epsilon>0}\ (s-\epsilon, s+\epsilon) \subseteq S \subseteq T$. Then $s \in \text{Int}(T)$ (again item 16).

   Proof #2: $\text{Int}(T)$ is the union of all open subsets of T, one of which is $\text{Int}(S) \subseteq S \subseteq T$.

4. Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Suppose that S is dense.
   Show that $S - \{x\}$ is also dense.

   Proof #1: Item 19(d) says that every non-empty open set contains infinitely many elements from S. If you delete x then there are still infinitely many left. So every non-empty open set contains infinitely many elements of $S - \{x\}$. Then $S - \{x\}$ is dense by 19(d).

   Proof #2: If we use the definition in item 19 then we are given to that **every** open interval $(\tilde{x}-\epsilon, \tilde{x}+\epsilon)$ contains an element of S. That is reworded in 19(c) to say that **every** non-empty open set contains an element of S (why is that equivalent? Well, because every non-empty open set contains an open interval!).

   Note: we do not know if the x in Ex 4 is the same as the x in item 19. We can not use the same symbol for things that might be different. Any time that happens, just use a different letter, or the same letter but with a prime x' or a tilde $\tilde{x}$ attached to it.

   If we want to prove "$S - \{x\}$ is dense" using the definition then have to show that every open interval $(\tilde{x}-\epsilon, \tilde{x}+\epsilon)$ contains an element of $S - \{x\}$. You can't just say: $(\tilde{x}-\epsilon, \tilde{x}+\epsilon) \bigcap S \neq \emptyset$ and $(\tilde{x}-\epsilon, \tilde{x}+\epsilon) \bigcap (\mathbb{R} - \{x\}) \neq \emptyset$ because just because two sets are non-empty, it doesn't imply that their intersection is non-empty too. So the key to the proof is to apply the given statement (that **every** non-empty open set contains an element of S) not to this open set: $(\tilde{x}-\epsilon, \tilde{x}+\epsilon)$ but to this open set: $(\tilde{x}-\epsilon, \tilde{x}+\epsilon) - \{x\}$. (or to any open interval in there).

   Proof #3: If we want to use item 19(b) then we have to show, for any $\alpha \in \mathbb{R}$, that there exists a sequence $a_1, a_2, \ldots \in S - \{x\}$ that converges to α. Make sure not to use the same letter for this α and this x because we do not know if they are the same. Now there are two cases:
   Case 1: $\alpha = x$. We could choose $a_n \in (x, x+1/n) \bigcap S$ (that intersection is non-empty because S is dense). Then $a_1, a_2, \ldots \in S - \{x\}$ and it converges to $\alpha = x$.
   Case 2: $\alpha \neq x$. Now we could take a sequence $a_1, a_2, \ldots$ in S that converges to α. Such a sequence exists because S is dense, use 19(b).

The problem now is that we do not know if this same sequence is also in $S - \{x\}$. To finish the proof, note that a **tail** of the sequence must be in $S - \{x\}$ (apply item 8 with $\epsilon := |\alpha - x|$).

5. Let $S \subseteq \mathbb{R}$ and $x \in \mathbb{R}$. Suppose that $S \bigcup \{x\}$ is open. Show that $x \in \overline{S}$.

   Proof #1: From item 3 we get $(x - \epsilon, x + \epsilon) \subseteq S \bigcup \{x\}$ for some $\epsilon > 0$. Then $(x - \epsilon, x) \bigcup (x, x + \epsilon)$ is a subset of S.
   But recall from Ex 1 that x is a limit point of $(x - \epsilon, x) \bigcup (x, x + \epsilon)$.
   Then x is also a limit point of S. Then use item 11(a).

   Proof #2: If you want to write a proof by contrapositive, you assume $x \notin \overline{S}$. By item 11(e) that is the same as saying $(x - \epsilon, x + \epsilon) \bigcap S = \emptyset$ for some $\epsilon > 0$. The To-Prove statement in the proof-by-contrapositive method is: T.P. $S \bigcup \{x\}$ is not open. That means: copy the definition of open from item 3, and negate it.
   Warning: there is no guarantee that the x from item 3 is the same as the x in Ex 5. It might be the same, but we do not (yet) know that so we must use a different letter! The same goes for the ϵ too. Then here is the negation of "$S \bigcup \{x\}$ open":
   $\exists_{\tilde{x} \in S \bigcup \{x\}} \forall_{\tilde{\epsilon} > 0} \ (\tilde{x} - \tilde{\epsilon}, \tilde{x} + \tilde{\epsilon}) \not\subseteq S \bigcup \{x\}$
   Proof: take $\tilde{x} := x$

   (How did I know to start like that? Well, WP#6 tells us to write "Take $\tilde{x} := \ldots$ but the only element in that set $S \bigcup \{x\}$ that I actually know is x so I don't see any other options than to write: take $\tilde{x} := x$.)

   None of the points in $(x - \epsilon, x + \epsilon)$ are in S, so it is not possible, for any $\tilde{\epsilon} > 0$, that all of the infinitely many points in $(\tilde{x} - \tilde{\epsilon}, \tilde{x} + \tilde{\epsilon}) = (x - \tilde{\epsilon}, x + \tilde{\epsilon})$ would be in $S \bigcup \{x\}$. So I conclude: $\forall_{\tilde{\epsilon} > 0} \ (\tilde{x} - \tilde{\epsilon}, \tilde{x} + \tilde{\epsilon}) \not\subseteq S \bigcup \{x\}$.

Good news: the quiz to add points to test 2 went very well.
Bad news: test 3 did not go well.
Good news: there is enough time to have a similar quiz for test 3.