3. If \( S \subseteq T \subseteq \mathbb{R} \), then show that \( \text{Int}(S) \subseteq \text{Int}(T) \).

Proof #1: Let \( s \in \text{Int}(S) \). To prove \( s \in \text{Int}(T) \).

The definition in item 16 says \( \exists \epsilon > 0 \) \((s - \epsilon, s + \epsilon) \subseteq S \). But \( S \subseteq T \) so \( \exists \epsilon > 0 \) \((s - \epsilon, s + \epsilon) \subseteq S \subseteq T \). Then \( s \in \text{Int}(T) \) (again item 16).

Proof #2: \( \text{Int}(T) \) is the union of all open subsets of \( T \), one of which is \( \text{Int}(S) \subseteq S \subseteq T \).

4. Let \( S \subseteq \mathbb{R} \) and \( x \in \mathbb{R} \). Suppose that \( S \) is dense.

Show that \( S - \{x\} \) is also dense.

Proof #1: Item 19(d) says that every non-empty open set contains infinitely many elements from \( S \). If you delete \( x \) then there are still infinitely many left. So every non-empty open set contains infinitely many elements of \( S - \{x\} \). Then \( S - \{x\} \) is dense by 19(d).

Proof #2: If we use the definition in item 19 then we are given to that every open interval \((\bar{x} - \epsilon, \bar{x} + \epsilon)\) contains an element of \( S \). That is reworded in 19(c) to say that every non-empty open set contains an element of \( S \) (why is that equivalent? Well, because every non-empty open set contains an open interval!).

Note: we do not know if the \( x \) in Ex 4 is the same as the \( x \) in item 19. We can not use the same symbol for things that might be different. Any time that happens, just use a different letter, or the same letter but with a prime \( x' \) or a tilde \( \bar{x} \) attached to it.

If we want to prove “\( S - \{x\} \) is dense” using the definition then have to show that every open interval \((\bar{x} - \epsilon, \bar{x} + \epsilon)\) contains an element of \( S - \{x\} \). You can’t just say: \((\bar{x} - \epsilon, \bar{x} + \epsilon) \cap S \neq \emptyset \) and \((\bar{x} - \epsilon, \bar{x} + \epsilon) \cap (\mathbb{R} - \{x\}) \neq \emptyset \) because just because two sets are non-empty, it doesn’t imply that their intersection is non-empty too. So the key to the proof is to apply the given statement (that every non-empty open set contains an element of \( S \)) not to this open set: \((\bar{x} - \epsilon, \bar{x} + \epsilon)\) but to this open set: \((\bar{x} - \epsilon, \bar{x} + \epsilon) - \{x\}\). (or to any open interval in there).

Proof #3: If we want to use item 19(b) then we have to show, for any \( \alpha \in \mathbb{R} \), that there exists a sequence \( a_1, a_2, \ldots \in S - \{x\} \) that converges to \( \alpha \). Make sure not to use the same letter for this \( \alpha \) and this \( x \) because we do not know if they are the same. Now there are two cases:

Case 1: \( \alpha = x \). We could choose \( a_n \in (x, x + 1/n) \cap S \) (that intersection is non-empty because \( S \) is dense). Then \( a_1, a_2, \ldots \in S - \{x\} \) and it converges to \( \alpha = x \).

Case 2: \( \alpha \neq x \). Now we could take a sequence \( a_1, a_2, \ldots \) in \( S \) that converges to \( \alpha \). Such a sequence exists because \( S \) is dense, use 19(b).
The problem now is that we do not know if this same sequence is also in \( S - \{ x \} \). To finish the proof, note that a tail of the sequence must be in \( S - \{ x \} \) (apply item 8 with \( \epsilon := |\alpha - x| \)).

5. Let \( S \subseteq \mathbb{R} \) and \( x \in \mathbb{R} \). Suppose that \( S \cup \{ x \} \) is open. Show that \( x \in \overline{S} \).

Proof #1: From item 3 we get \((x - \epsilon, x + \epsilon) \subseteq S \cup \{ x \}\) for some \( \epsilon > 0 \). Then \((x - \epsilon, x) \cup (x, x + \epsilon)\) is a subset of \( S \).

But recall from Ex 1 that \( x \) is a limit point of \((x - \epsilon, x) \cup (x, x + \epsilon)\).
Then \( x \) is also a limit point of \( S \). Then use item 11(a).

Proof #2: If you want to write a proof by contrapositive, you assume \( x \not\in \overline{S} \). By item 11(e) that is the same as saying \((x - \epsilon, x + \epsilon) \cap S = \emptyset\) for some \( \epsilon > 0 \). The To-Prove statement in the proof-by-contrapositive method is: T.P. \( S \cup \{ x \} \) is not open. That means: copy the definition of open from item 3, and negate it.

Warning: there is no guarantee that the \( x \) from item 3 is the same as the \( x \) in Ex 5. It might be the same, but we do not (yet) know that so we must use a different letter! The same goes for the \( \epsilon \) too. Then here is the negation of “\( S \cup \{ x \} \) open”:

\[ \exists \tilde{x} \in S \cup \{ x \} \forall \tilde{\epsilon} > 0 (\tilde{x} - \tilde{\epsilon}, \tilde{x} + \tilde{\epsilon}) \not\subseteq S \cup \{ x \} \]

Proof: take \( \tilde{x} := x \)

(How did I know to start like that? Well, WP#6 tells us to write “Take \( \tilde{x} := \ldots \) but the only element in that set \( S \cup \{ x \} \) that I actually know is \( x \) so I don’t see any other options than to write: take \( \tilde{x} := x \).

None of the points in \((x - \epsilon, x + \epsilon)\) are in \( S \), so it is not possible, for any \( \tilde{\epsilon} > 0 \), that all of the infinitely many points in \((\tilde{x} - \tilde{\epsilon}, \tilde{x} + \tilde{\epsilon}) = (x - \epsilon, x + \epsilon)\) would be in \( S \cup \{ x \} \). So I conclude: \( \forall \tilde{\epsilon} > 0 (\tilde{x} - \tilde{\epsilon}, \tilde{x} + \tilde{\epsilon}) \not\subseteq S \cup \{ x \} \).

Good news: the quiz to add points to test 2 went very well.
Bad news: test 3 did not go well.
Good news: there is enough time to have a similar quiz for test 3.