GRV I, answers test 1, Monday September 28, 2020.

1. Let σ be the following element of S_{10}.

$$\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7)(6\ 7\ 8\ 9\ 10).$$

   Compute a disjoint cycle notation for this permutation σ: $(1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10)$.

   Compute the order of σ: lcm(6,4) = 12.

2. Let G be a group with n elements, and suppose that $\gcd(k, n) = 1$. Prove that for every $g \in G$ there exists $h \in G$ for which $h^k = g$.

   $|G| = n$ and so $g^n = e$ for any $g \in G$. Bezout's equation says $sk + tn = 1$ for some integers s, t. Then $g = g^1 = g^{sk+tn} = g^{sk} g^{tn} = (g^s)^k (g^n)^t = (g^s)^k e = h^k$ where $h := g^s$.

3. Suppose that a, b, and ab each have order 2. Show that a and b commute.

   ab has order 2, so $abab = (ab)^2 = e$. Since a and b also have order 2, we have $aabb = e$. So $abab = aabb$. By the cancellation law, we can cancel the left-a and right-b and get $ba = ab$.

4. Let G be a group and let H be a subgroup of index 2 (so G/H has 2 elements, H and $G - H$). Show that H is a normal subgroup of G.

   The left-cosets form a partition, and the index of H is the number of parts in this partition, so there are two parts. Since H is one of those parts, the other part in this partition can only be $G - H$. Thus $G/H = \{\text{left cosets}\} = \{H, G - H\}$. For the same reason, the right-cosets are also H and $G - H$. But then every left-coset is a right-coset, and thus H is a normal subgroup.

   Proof #2: Let $A := G/H$. The group G acts on A by left-multiplication, this action is not trivial, and thus this gives a non-trivial homomorphism from G to $S_A \cong S_2 \cong Z_2$. If K is the kernel, then $G/K \cong Z_2$ and thus K has index 2. If $k \in K$ then $ka = a$ for each $a \in A$, including $a = H$, and thus $k \in H$, proving that $K \leq H \leq G$. But K and H both have index 2 in G, so the index of K in H must be 1, and so $K = H$.

5. Let G is a group. Let $A = G$ and let G act on A by conjugation, so $g \cdot a = gag^{-1}$. Now fix one element $a \in A$. Let $\mathcal{O} := \{g \cdot a \mid g \in G\} = \{gag^{-1} \mid g \in G\}$.

   (a) If $g_1 \cdot a = g_2 \cdot a$ then show that $g_1 \in g_2\, C_G(a)$.

   Assume $g_1 \cdot a = g_2 \cdot a$. Then $g_2^{-1} g_1 \cdot a = a$. By the definition of $\cdot$ this means $g_2^{-1} g_1$ commutes with a, so $g_2^{-1} g_1 \in C_G(a)$. Multiplying by g_2 this becomes: $g_1 \in g_2\, C_G(a)$.

   Remark: the converse is true as well, if $g_1 \in g_2\, C_G(a)$ then $g_1 = g_2 c$ for some $c \in C_G(a)$, but then $c \cdot a = a$ and thus $g_1 \cdot a = g_2 c \cdot a = g_2 \cdot a$.

   (b) Show $|\mathcal{O}| = |G : C_G(a)|$.

   The map $G \to \mathcal{O}$ given by $g \mapsto g \cdot a$ is surjective, and by part (a), the fibers of this map are the left-cosets of $C_G(a)$. The number of elements of $\mathcal{O}$ is the number of fibers, which is the number of left-cosets, which is $|G : C_G(a)|$.