Finding all Bessel type solutions for Linear Differential Equations with Rational Function Coefficients

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ABSTRACT

A linear differential equation with rational function coefficients has a Bessel type solution when it is solvable in terms of $B_\nu(f), B_{\nu+1}(f)$. For second order equations, with rational function coefficients, f must be a rational function or the square root of a rational function. An algorithm was given by Debeerst, van Hoeij, and Koepf, that can compute Bessel type solutions if and only if f is a rational function. In this paper we extend this work to the square root case, resulting in a complete algorithm to find all Bessel type solutions.

1. INTRODUCTION

Let $a_0, a_1, a_2 \in \mathbb{C}(x)$ and let $L = a_2\partial^2 + a_1\partial + a_0$ be a differential operator of order two. The corresponding differential equation is $L(y) = 0$, i.e. $a_2y'' + a_1y' + a_0y = 0$. Let $B_\nu(x)$ denote one of the Bessel functions (one of Bessel I, J, K, or Y functions). The question studied in [6, 7] is the following: Given L, decide if there exists a rational function $f \in \mathbb{C}(x)$ such that L has a solution y that can be expressed[1] in terms of $B_\nu(f)$. If so, then find f, ν, and the corresponding solutions of L. The same problem was also solved for Kummer/Whittaker functions, see [6]. This means that for second order L, with rational function coefficients, there is an almost-complete algorithm in [6] to decide if $L(y) = 0$ is solvable in terms of ${}_0F_1$ or ${}_1F_1$ functions, and if so, to find the solutions.

The reason this almost-complete algorithm is not complete is the following: If $B_\nu(f)$ satisfies a second order linear differential equation with rational function coefficients, then either: $f \in \mathbb{C}(x)$, or (*square root case*): $f \notin \mathbb{C}(x)$ but $f^2 \in \mathbb{C}(x)$.

However, only the $f \in \mathbb{C}(x)$ case was handled in [6, 7], the square-root case was listed in the conclusion of [7] as a task for future work. This meant that [6, 7] is not yet a complete solver for ${}_0F_1$ and ${}_1F_1$ type solutions.

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[1]using sums, products, differentiation, and exponential integrals (see Definition 2)

In this paper, we treat the square-root case for Bessel functions. The combination of this paper with the treatment of Kummer/Whittaker functions in [6] is then a complete algorithm to find ${}_0F_1$ and ${}_1F_1$ type solutions whenever they exist[2].

The reason why the square-root case was not yet treated in [7] will be explained in the next two paragraphs. If f is a rational function $f = A/B$, then from the generalized exponents at the irregular singularities, we can compute B, as well as $\deg(A)$ linear equations for the coefficients of A, see [7], or see [6] which contains more details and examples. Since a polynomial A of degree $\deg(A)$ has $\deg(A) + 1$ coefficients, this meant that only one more equation was needed to reconstruct A, and in each of the various cases in [6, 7] there was a way to compute such an equation.

In the square-root case, we can not write f as a quotient of polynomials, but we can write $f^2 = A/B$. The same method as in [6, 7] will still produce B, and linear equations for the coefficients of A. The number of linear equations for the coefficients of A is still the same as it was in the $f \in \mathbb{C}(x)$ case. Unfortunately, by squaring f to make it a rational function, we doubled the degree of A, but we do not get more linear equations, which means that in the square-root case the number of linear equations is only $\frac{1}{2}\deg(A)$ (plus an additional ≥ 0 equations coming from regular singularities). So in the worst case, the number of equations is only half of the degree of A. This is why the square-root case was not solved in [7] but only mentioned as a future task.

Our approach is the following: One can rewrite $A = CA_1A_2^d$ where A_1 can be computed from the regular singularities, but A_2 can not. The problem is that while the degree of A_2 is only $\frac{1}{d}$ times the degree of A/A_1, the linear equations on the coefficients of A translate into polynomial equations (with degree d) for the coefficients of A_2. Solving systems of polynomial equations can take too much CPU time. However, we discovered that with some modifications, one can actually obtain linear equations for the coefficients of A_2. This means that we only need to solve linear systems. The result is an efficient algorithm that can handle complicated inputs. An implementation is available online at http://www.math.fsu.edu/~qyuan.

2. PRELIMINARIES

[2]Other ${}_0F_1$ and ${}_1F_1$ type functions can be rewritten in terms of Bessel, or Kummer/Whittaker functions. For instance, Airy type functions form a subclass of Bessel type functions (provided that the square-root case is treated!)

2.1 Differential Operators

We let $K[\partial]$ be the ring of differential operators with coefficients in a differential field K. Let C_K be the field of constant, $\overline{C_K}$ be algebraic closure of C_K. Usually, we have $K = C_K(x)$ and C_K is finite extension of $\mathbb{Q}$.

We call $p \in \overline{C_K} \cup \{\infty\}$ a *singularity* of differential operator $L \in K[\partial]$, if $p = \infty$ or p is a zero of the leading coefficient of L or p is a pole of a coefficient of L. If p is not a singularity, p is *regular*.

We say y is a solution of L, if $L(y) = 0$. The vector space of solutions is denoted as $V(L)$. If p is regular, we can express all solutions around p as power series $\sum_{i=0}^{\infty} b_i t_p^i$ where t_p denotes the local parameter which is $t_p = \frac{1}{x}$ if $p = \infty$ and $t_p = x - p$, otherwise.

2.2 Formal Solutions and Generalized Exponents

Definition 1. *We say $e \in \mathbb{C}[t_p^{-\frac{1}{m}}]$ is a generalized exponent of L at p if and only if L has a solution $\exp\left(\int \frac{e}{t_p}\, dt_p\right) S$, $S \in R_m$, and $S \notin t_p^{\frac{1}{m}} R_m$, where $R_m = \mathbb{C}[[t_p^{\frac{1}{m}}]][\log(t_p)]$*

If $e \in \mathbb{C}$ we just get a solution $x^e S$, in this case e is called an *exponent*. If the solution involves a logarithm, we call it a *logarithmic solution*. If $m = 1$, then e is *unramified*, otherwise it is *ramified*.

Remark 1. *Since we only consider second order differential operators, m in the definition can be only 1 or 2.*

If the order of L is n, then at every point p, counting with multiplicity, there are n generalized exponents $e_1, e_2, ..., e_n$, and the solutions $\exp\left(\int \frac{e_i}{t_p}\, dt_p\right) S_i$, $i = 1, ..., n$ are a basis of solution space $V(L)$. If p is regular, then the generalized exponents of L at p are $0, 1, ..., n-1$. One can compute generalized exponents with the Maple command DEtools[gen_exp].

2.3 Bessel Functions

Bessel functions are the solutions of the operators $L_{B1} = x^2\partial^2 + x\partial + (x^2 - \nu^2)$ and $L_{B2} = x^2\partial^2 + x\partial - (x^2 + \nu^2)$. The two linearly independent solutions $J_\nu(x)$ and $Y_\nu(x)$ of L_{B1} are called Bessel functions of first and second kind, respectively. Similarly the solutions $I_\nu(x)$ and $K_\nu(x)$ of L_{B2} are called the modified Bessel functions of first and second kind. Let $B_\nu(x)$ refer to one of the Bessel functions.

When ν is half integer, L_{B1} and L_{B2} are reducible. One can get the solutions by factoring the operators. We will exclude this case from this paper.

The change of variables $x \to ix$ sends $V(L_{B1})$ to $V(L_{B2})$ and vice versa. Since our algorithm will deal with change of variables, as well as two other transformations (see Section 2.4), we only need one of L_{B1}, L_{B2}. We choose L_{B2} and denote $L_B := L_{B2}$.

L_B has only two singularities, 0 and ∞. The generalized exponents are $\pm\nu$ at 0 and $\pm t_\infty^{-1} + \frac{1}{2}$ at ∞.

After a change of variables $y(x) \to y(\sqrt{x})$, we get a new operator $L_B^{\surd} = x^2\partial^2 + x\partial - \frac{1}{4}(x + \nu^2)$, which is still in $\mathbb{Q}(x)[\partial]$. Let $\mathrm{CV}(L, f)$ denote the operator obtained from L by change of variables $x \mapsto f$. For any differential field extension K of $\mathbb{Q}(x)$, if $\nu^2 \in C_K$, and if $f^2 \in K$, then $\mathrm{CV}(L_B, f) \in K[\partial]$ since this operator can can be written as $\mathrm{CV}(L_B^{\surd}, f^2)$. The converse is also true:

Lemma 1. *Let K be a differential field extension of $\mathbb{Q}(x)$, let f, ν be elements of a differential field extension of K, and ν be constant. Then*

$$\mathrm{CV}(L_B, f) \in K[\partial] \iff f^2 \in K \text{ and } \nu^2 \in C_K.$$

Proof. It remains to prove $\Longrightarrow$. Let ν be a constant and

$$M := \mathrm{monic}(\mathrm{CV}(L_B, f)) = \partial^2 + a_1\partial + a_0$$

We have to prove

$$a_0, a_1 \in K \Longrightarrow f^2, \nu^2 \in K$$

and so we assume $a_0, a_1 \in K$. Let $g = f^2$. By computing $M = \mathrm{monic}(\mathrm{CV}(L_B^{\surd}, g))$ we find

$$a_1 = -\mathrm{ld}(\mathrm{ld}(g)), \quad a_0 = \frac{-1}{4}(g + \nu^2)\mathrm{ld}(g)^2$$

where ld denotes the logarithmic derivative, $\mathrm{ld}(a) = a'/a$. Let

$$a_2 := \mathrm{ld}(\mathrm{ld}(a_0) + 2a_1) + \mathrm{ld}(a_0) + 3a_1,$$

$$a_3 := -4a_0/a_2^2, \quad a_4 := a_3(2a_1 + \mathrm{ld}(a_0))$$

which are in K since $a_0, a_1 \in K$. Direct substitution shows that $a_2 = \mathrm{ld}(g)$, $a_3 = g + \nu^2$, and $a_4 = g'$. Hence $g = a_4/a_2 \in K$ and $\nu^2 = a_3 - g \in K$. $\square$

2.4 Transformations

Definition 2. *A transformation between two differential operators L_1 and L_2 is an onto map from solution space $V(L_1)$ to $V(L_2)$. For an order 2 operator $L_1 \in K[\partial]$, there are three types of transformations for which the resulting L_2 is again in $K[\partial]$ with order 2. They are (notation as in [6, 7]):*

(i) change of variables: $y(x) \to y(f(x))$, $f(x) \in K \setminus C_K$.
(ii) exp-product: $y \to \exp(\int r\, dx) \cdot y$, $r \in K$.
(iii) gauge transformation: $y \to r_0 y + r_1 y'$, $r_0, r_1 \in K$.

We denote it by $\longrightarrow_C$, $\longrightarrow_E$, $\longrightarrow_G$ respectively.

We can switch the order of $\longrightarrow_E$ and $\longrightarrow_G$ [6]. So we will denote $L_1 \longrightarrow_{EG} L_2$ if some combination of (ii) and (iii) sends L_1 to L_2. Likewise we denote $L_1 \longrightarrow_{CEG} L_2$ if some combination of (i), (ii),(iii) sends L_1 to L_2.

Remark 2. *The relation $\longrightarrow_{EG}$ is an equivalence relation. But $\longrightarrow_C$ is not.*

Definition 3. *We say $L_1 \in K[\partial]$ is projectively equivalent to L_2 if and only if $L_1 \longrightarrow_{EG} L_2$.*

Lemma 2. *(Lemma 3 in [7]) If $L_1 \longrightarrow_{CEG} L_2$, then there exist an operator $M \in K[\partial]$ such that $L_1 \longrightarrow_C M \longrightarrow_{EG} L_2$*

We can apply Lemma 2 to $L_1 = L_B^{\surd}$ and L_2 which is the operator L we want to solve. If M is known (i.e if the change of variables $x \to f$ is known), then the map from $V(M)$ to $V(L)$ can be computed with existing algorithms [1], [6]. That means $L_B^{\surd} \longrightarrow_{CEG} L$ can be computed if we can find the change of variables.

The goal of this paper is to solve differential equations in terms of Bessel functions. This means: if $L_B^{\surd} \longrightarrow_{CEG} L$, then solve L.

Main problem: Let C_K be a field, $C_K \subseteq \mathbb{C}$, and let $K = C_K(x)$. Let $L \in K[\partial]$ be irreducible and of order 2. The question we will solve in this paper is the following: Does there exist an operator $M \in K[\partial]$ such that

1. L is projectively equivalent to M, and
2. $L_B^{\surd} \xrightarrow{g}_C M$ for some $g \in K$ and some constant ν.

If so, find g, ν and solve L.

Note that $L_B^{\surd} \xrightarrow{g}_C M$ is the same as $L_B \xrightarrow{f}_C M$, where $f^2 = g \in K$. The reason we use the second form is because we can then use the same notation as in [6] [7].

3. THE CHANGE OF VARIABLES

3.1 The Exponent Difference

To recover f in the transformation $\longrightarrow_C$, we need some information about f and this information should be invariant under projective equivalence. Since the order of L is 2, we have two exponents e_1, e_2 at a point p. We consider the *exponent difference* $\Delta(L,p) = \pm(e_1 - e_2)$. One can verify that Δ modulo $\frac{1}{m}\mathbb{Z}$ is invariant under projective equivalence [6] (Here m is as in Section 2.2). We use a $\pm$ sign because we don't know the order of the generalized exponents. We also define:

Definition 4. *A singularity p of $L \in K[\partial]$ is called:*

(i) apparent singularity if and only if $\Delta(L,p) \in \mathbb{Z}$ and L is not logarithmic at p. This is equivalent to saying that L has a basis of solutions y_1, y_2 for which y_1/y_2 is analytic at p.

(ii) regular singularity if and only if $\Delta(L,p) \in \mathbb{C} \setminus \mathbb{Z}$ or L is logarithmic at p.

(iii) irregular singularity if and only if $\Delta(L,p) \in \mathbb{C}[t_p^{-\frac{1}{2}}] \setminus \mathbb{C}$

we also denote the set of regular singularities and irregular singularities by S_{reg} and S_{irr}.

Note: Apparent singularities are not in S_{reg} nor S_{irr}.

The main work in this paper is to construct a finite set of candidates for (f, ν) from the $\Delta(L,p)$. Let $g = f^2 \in K$. If g has a root resp. pole at p of order $k \in \mathbb{N}$, then we say that f has a root resp. pole at p of order $m_p := \frac{k}{2}$.

Theorem 1. *Let $K = C_K(x)$, and $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$, where $f^2 \in K$. Note: L is the input to our algorithm, and f and M are to be computed.*

(i) if p is a zero of f with multiplicity $m_p \in \frac{1}{2}\mathbb{Z}^+$, then p is an apparent singularity or $p \in S_{reg}$, and $\Delta(M,p) = 2m_p\nu$.

(ii) p is a pole of f with pole order $m_p \in \frac{1}{2}\mathbb{Z}^+$ such that $f = \sum_{i=-m_p}^{\infty} f_i t_p^i$, if and only if $p \in S_{irr}$ and $\Delta(M,p) = 2\sum_{i<0} i f_i t_p^i$.

If $p \in S_{reg}$, then $\Delta(L,p) \equiv \Delta(M,p)$ mod $\mathbb{Z}$ which means that we can compute $2m_p\nu$ mod $\mathbb{Z}$.
If $p \in S_{irr}$, then $\Delta(L,p) \equiv \Delta(M,p)$ mod $\frac{1}{m}\mathbb{Z}$. Then $\sum_{i<0} f_i t_p^i$ can be computed from $\Delta(L,p)$ by dividing coefficients by $2i$ (the congruence only affects the t_p^0-term of Δ, but that term is not used when $p \in S_{irr}$).

Proof. We can use the same proof in [6]. □

Definition 5. *Let $f = \Sigma_{i=N}^{\infty}$, $N \in \mathbb{Z}$, $a_N \neq 0$. We say that we have a k-term truncated power series for f when the coefficient of $x^N, ..., x^{N+k-1}$ are known.*

Remark 3. *If a k-term truncated series for f is known, then we can compute a k-term truncated series for f^2.*

According to Theorem 1 (ii), from $\Delta(M,p)$, we can get a $\lceil m_p \rceil$-term truncated series of f at p. In [7], f was assumed to be in K, in which case the truncated series is exactly the polar part of f at p. But in this paper, we have to compute $g = f^2 \in K$. Theorem 1 (ii) gives us the polar part of f, i.e. a truncated series for f. We square it to obtain a truncated series of g. But this truncated series for g has $\lceil m_p \rceil$ terms (the same number of terms as the one for f, see Remark 3). So it is only half (rounded up) of the polar part of g. For instance, if f has a pole of order 3 at $x = 0$, from $\Delta(L,p)$ we can obtain a truncated series $\Sigma_{i=-3}^{-1} a_i x^i$ of f at 0. Squaring this series, we can get the coefficients of x^{-6}, x^{-5}, x^{-4} of g, but not more. So we have:

Corollary 1. *If $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$ and $g = f^2$ then we have:*

(i) if $p \in S_{reg}$ then p is a zero of g.

(ii) $p \in S_{irr}$ if and only if p is a pole of g. We can also get a $\lceil m_p \rceil$-term truncated series of g from $\Delta(L,p)$, where m_p is the pole order of f.

3.2 The Parameter ν

The exponent difference is also associated with the Bessel parameter ν. In [6], we have:

Theorem 2. *If $L_B^{\surd} \longrightarrow_{CEG} L$, then*

(i) if $S_{reg} = \emptyset$ then $\nu \in \mathbb{Q} \setminus \mathbb{Z}$.

The following hold for any $p \in S_{reg}$:

(ii) L logarithmic at p if and only if $\nu \in \mathbb{Z}$.

(iii) if $\Delta(L,p) \in \mathbb{Q}$ then $\nu \in \mathbb{Q} \setminus \mathbb{Z}$.

(iv) $\Delta(L,p) \in C_K \setminus \mathbb{Q}$ if and only if $\nu \in C_K \setminus \mathbb{Q}$.

(v) $\Delta(L,p) \notin C_K$ if and only if $\nu \notin C_K$.

We will divide our algorithm into different cases by different situations in Theorem 2. We call (ii) *logarithmic case*. (i) and (iii) *rational case*, and (iv) and (v) *irrational case*. We also have *easy case* which will be defined later. For the logarithmic case, we have

Remark 4. *If any $p \in S_{reg}$ is logarithmic then by Theorem 2 (ii), $\nu \in \mathbb{Z}$, then by again Theorem 2 (ii), we have every $p \in S_{reg}$ must be logarithmic. If not, then L has no Bessel type solutions. Also by the fact $\mathbb{C}(x)B_\nu(x) + \mathbb{C}(x)B'_\nu(x)$ is invariant under $\nu \to \nu + 1$ and $\nu \to 1 - \nu$, for the logarithmic case, we can let $\nu = 0$.*

For the rational case:

Remark 5. *Since $\mathbb{C}(x)B_\nu(x) + \mathbb{C}(x)B'_\nu(x)$ is invariant under $\nu \to \nu + 1$ and $\nu \to 1 - \nu$. for $\nu \in \mathbb{Q}$, we can just focus on $\nu \in [0, \frac{1}{2}]$. Also since when $\nu = \frac{1}{2}$ the operator will be reducible, it is easy to solve the operator by factoring. So we just consider $\nu \in [0, \frac{1}{2})$.*

We will give a method to find a finite list of candidates for ν and f later. If we fix f, then we have:

LEMMA 3. *Let Z be the set of all zeroes of f, for $p \in Z$ let m_p be the multiplicity at p.*

(i) If $\Delta(L,p) \in C_K$, then let

$$\mathcal{N}'_p := \left\{ \frac{\Delta(L,p)+i}{2m_p} \mid 0 \le i \le 2m_p - 1, i \in \mathbb{Z} \right\}$$

. We can make the rational part of each element in $\mathcal{N}'_p$ belong to $[0, \frac{1}{2}]$. Let the new set be $\mathcal{N}_p$. Then $\nu \in \mathcal{N} := \cap_{p \in S_{reg}} \mathcal{N}_p$.

(ii) If $\Delta(L,p) \notin C_K$, we can write $\Delta(L,p)$ as $a_1\sqrt{k} + a_2$ where $k \in C_K$ and a_1, $a_2 \in C_K$. Then $\nu = \frac{a_1\sqrt{k}}{2m_p}$ (if for different p, we get different ν then there are no Bessel type solutions.)

PROOF. The lemma follows from the fact that we know the number $\Delta(M,p) = 2m_p\nu \bmod \mathbb{Z}$, the fact that $\nu^2 \in C_K$, and the fact that $\mathbb{C}(x)B_\nu(x) + \mathbb{C}(x)B'_\nu(x)$ is invariant under $\nu \to \nu + 1$ and $\nu \to 1 - \nu$. □

3.3 Easy, Logarithmic and Irrational Cases

To retrieve f, we need enough linear equations. We assume $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$. We want to get information about f from L. Since f might not in K, but $g = f^2$ is in K, we can assume $g = \frac{A}{B}$, $A, B \in C_K[x]$, B is monic and $\gcd(A,B) = 1$. We want to get information about A, B from L. Since Maple can compute generalized exponents of L, we can compute the exponent difference at singularity p. Then by Corollary 1 we can get the set S_{reg}, which give us some zeroes of g, and S_{irr}, which give the truncated series at each $p \in S_{irr}$. The following two lemmas are true for all cases:

LEMMA 4. *We can retrieve B from S_{irr}.*

PROOF. According to Theorem 1 (ii), if $p \in S_{irr}$ then p is a pole of f. Let $m_p \in \frac{1}{2}\mathbb{Z}^+$ be pole order of $\Delta(M,p)$. g has a pole order $2m_p$. The Theorem implies $B = \prod_{p \in S_{irr} \setminus \{\infty\}} (x - p)^{2m_p}$. □

LEMMA 5. *Let*

$$d_A = \begin{cases} \deg(B) + 2m_\infty & \text{if } \infty \in S_{irr} \\ \deg(B) & \text{otherwise} \end{cases}$$

(i) If $\infty \in S_{reg}$ then $\deg(A) < d_A$;

(ii) if $\infty \in S_{irr}$ then $\deg(A) = d_A$;

(iii) otherwise $\deg(A) \le d_A$.

In all cases, we can write $A = \Sigma_{i=0}^{d_A} a_i x^i$, so we have $d_A + 1$ unknowns.

PROOF. According to Corollary 1 (i), if $\infty \in S_{reg}$ then we have $\deg(A) < \deg(B)$. If $\infty \in S_{irr}$ with pole order m_∞, then $\deg(A) = \deg(B) + 2m_\infty$ (see Corollary 1 (ii)). If $\infty \notin S_{irr}$ then f does not have a pole at ∞, so that $\deg(A) \le \deg(B)$. □

LEMMA 6. *Assume $p \in C_K$, if $p \in S_{reg}$, we will get one linear equation for the coefficients of A. If $p \in S_{irr}$ with m_p as pole order of $\Delta(L,p)$, we will get $\lceil m_p \rceil$ linear equations.*

PROOF. According to Corollary 1 (i), if $p \in S_{reg}$, p is a zero of A. Then we will get a linear equation of$\{a_i\}_{i=0,\ldots,d_A}$ by setting $\operatorname{rem}(A, x-p) = 0$.
In addition, for each $p \in S_{irr}$ with with pole order m_p, by Corollary 1 (ii) we will have a $\lceil m_p \rceil$-term truncated series of g at p. Then we can get the truncated series of $A = gB$. On the other hand, we can rewrite $A = \Sigma_{i=0}^{d_A} a_i x^i$ as a truncated series at p (by Taylor or Laurent series). Since the terms in a Taylor series or Laurent series depend linearly on the coefficients of A, by comparing the coefficients, each term will give a linear equation of a_i. □

EXAMPLE 1. [3]

$$L = \partial^2 + \frac{10x^3 - 21x^2 + 12x - 4}{x(x-1)(x-2)(5x-2)}\partial - \frac{1}{36}\frac{28x^4 - 89x^3 + 105x^2 - 59x + 16)(5x-2)^2}{x^2(x-1)^2(x-2)^6}$$

, $K = \mathbb{Q}(x)$. Then we can compute $S_{reg} = \{0\}$, $S_{irr} = \{2\}$ and the truncated series of g at $x = 2$ is $6t_2^{-4} + 21t_2^{-3} + O(t_2^{-2})$, so $B = (x-2)^4$ and $d_A = 4$. We assume $A = \Sigma_{i=0}^4 a_i x^i$. Then $\operatorname{rem}(A,x) = a_0 = 0$ give us one linear equation. And since we can rewrite $\frac{A}{B} = (a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4)t_2^{-4} + (a_1 + 4a_2 + 12a_3 + 32a_4)t_2^{-3} + O(t_2^{-2})$, By comparing the coefficient of two truncated series, we can get 2 linear equations $a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4 = 6$ and $a_1 + 4a_2 + 12a_3 + 32a_4 = 21$. For this example, we have 5 unknowns and we only have 3 linear equations. But we can still solve it(see Example 3).

For $p \in \overline{C_K}$, $p \notin C_K$, we have:

LEMMA 7. *If $p \notin C_K$, let $l(x)$ be minimum polynomial of p over C_K. If $p \in S_{reg}$, we will have $\deg(l)$ linear equations. If $p \in S_{irr}$, we will have $\deg(l) \cdot \lceil m_p \rceil$ linear equations.*

PROOF. Since $p \notin C_K$, let $l(x)$ be minimum polynomial of p over C_K. If $p \in S_{reg}$, then p is a zero of g. Then all the conjugates in $\overline{C_K}$ of p are zeroes of g. There are $\deg(l)$ conjugated zeroes and by setting $\operatorname{rem}(A, l(x)) = 0$, we will get $\deg(l)$ linear equations with coefficient in C_K.

If $p \in S_{irr}$ with m_p as pole order of $\Delta(L,p)$, we can first consider it in the field $C_K(p)$. Then according to lemma 6 we will get $\lceil m_p \rceil$ linear equations with coefficients in $C_K(p)$. Let $c + \sum_{i=0}^n c_i a_i = 0$ be such an equation, where $\{a_i\}$ are unknowns and $\{c_i\}$ are coefficients in $C_K(p)$. We can rewrite the equation as $\sum_{i=0}^{\deg(l)-1} e_i p^i = 0$ where $\{e_i\}$ are linear functions with coefficients in C_K. Now p is algebraic over C_K of degree $\deg(l)$, so $1, p, ..., p^{\deg(l)-1}$ are linearly independent over C_K. Hence the e_i are 0; we get $\deg(l)$ linear equations over C_K. We can do this for each of the $\lceil m_p \rceil$ linear equations. Then we get $\deg(l) \cdot \lceil m_p \rceil$ equations. □

EXAMPLE 2. *Suppose $\sqrt{2} \in S_{reg}$. If $\sqrt{2} \in C_K$, we get one equation by $\operatorname{rem}(A, x - \sqrt{2}) = 0$. If $\sqrt{2} \notin C_K$, we get two equations by $\operatorname{rem}(A, x^2 - 2) = 0$.
Suppose $\sqrt{2} \in S_{irr}$, and that one of the $\lceil m_{\sqrt{2}} \rceil$ linear equations is $3 + (1-\sqrt{2})a_1 + (1+\sqrt{2})a_2 = 0$. If $\sqrt{2} \notin C_K$, we can rewrite that equation as $(3 + a_1 + a_2) + (a_2 - a_1)\sqrt{2} = 0$. Then we can get two equations $\{3 + a_1 + a_2 = 0, a_2 - a_1 = 0\}$.*

So far, we get at least $\#S_{reg} + \frac{1}{2}d_A$ linear equations for the coefficients of A. If this number is greater than $\deg(A)$, then we can solve them and find A. We call this case the *easy case*. This case is very similar to the case in [7].

For logarithmic case and irrational case, we have:

[3] the data is from examples at http://www.math.fsu.edu/~qyuan

LEMMA 8. *In both logarithmic and irrational case, we know all zeroes of A. In the irrational case, we know their multiplicities as well.*

PROOF. By Theorem 1 (i), a change of variables can transfer a regular singularity to an apparent singularity only if $\nu \in \mathbb{Q} \setminus \mathbb{Z}$. So in the logarithmic and irrational cases, S_{reg} contains all zeroes.
In the irrational case, for each $p \in S_{reg}$, let a_p be the coefficient of the irrational part of the exponent difference. Then there exists k, such that $k\Sigma_{p\in S_{reg}} a_p = d_A$. Then $\frac{a_p}{k}$ will give the multiplicity of p. □

In the irrational case, there is only one unknown coefficient, the leading coefficient of A. But we have $\frac{1}{2}d_A$ linear equations, enough to get A.

In the logarithmic case, we have to do a combinatorial search: try all possible combinations of multiplicities of zeroes of A. After that the only unknown is also leading coefficient of A. We have enough equations to find it.

Once we get f, we can get a list of ν by Lemma 3 and Remark 4.

3.4 Rational Case

The hardest case is the rational case. We will compute (see Lemma 9 and 10) a finite set of possible values for $d = \text{denom}(\nu)$. We notice that $d > 2$ because the case $\nu \in \mathbb{Z}$ has already been treated (logarithmic case) and if $\nu \in \frac{1}{2}\mathbb{Z}$ then L_B is reducible. Let $L_B \xrightarrow{f}_C M \longrightarrow_{EG} L$, $g = f^2 = \frac{A}{B}$. Let p be a root of A and $\Delta(L,p) \equiv 2m_p\nu \mod \mathbb{Z}$. If $d \mid 2m_p$, change of variables $x \mapsto f$ will send p to an apparent singularity. This is hard because if p is apparent, then $p \notin S_{reg}$, which means that not all roots of A are known (not all roots of A are in S_{reg}). But if a zero p of A becomes an apparent singularity, the multiplicity $2m_p$ [4] must be a multiple of d. So we can rewrite $A = CA_1A_2^d$, where $A_1, A_2 \in C_K[x]$ and $C \in C_K$, A_1 is monic and the roots of A_1 are the known roots of A (the elements of S_{reg}).

For $S_{reg} = \emptyset$, we can let $A_1 = 1$ and fix d by the following lemma [7]:

LEMMA 9. *If $S_{reg} = \emptyset$, then $d \mid d_A$.*

For $S_{reg} \neq \emptyset$, we have:

LEMMA 10. *If $S_{reg} \neq \emptyset$, we can find a list of candidate pairs (d, A_1) by solving an equation.*

PROOF. we assume $N = \#S_{reg}$, $S_{reg} = \{p_1, ..., p_N\}$ and $\Delta(L,p)$ is the exponent difference at p. Let $A_1 = \Pi_{i=1}^N (x - p_i)^{m_{p_i}}$, $1 \le m_{p_i} < d$ and $d_p = \text{denom}(\Delta(L,p))$. For each point $p \in S_{reg}$, $d_p \mid d$. So we have $l \mid d$ where $l := \text{lcm}_{p\in S_{reg}} d_p$. So d can only be a multiple of l, and it must be $\le d_A$. So there are $\lfloor d_A/l \rfloor$ possibilities for d. Once we fix d, then for each $p \in S_{irr}$ we have $\frac{d}{d_p} \mid m_p$. So solve $(\sum_{i=1}^N m_{p_i}) + \deg(A_2)d = d_A$, $1 \le m_{p_i} < d$ and $\frac{d}{d_{p_i}} \mid m_{p_i}$. It will give finitely many candidates for A_1. □

LEMMA 11. *We can choose C, such that the dth root of the coefficient of the initial term of the truncated series of $A/(CA_1)$ at p is in C_K*

[4] If m_p is multiplicity of f at p, then $2m_p$ is multiplicity of A.

PROOF. If $(C_K \cup \infty) \cap S_{irr} = \emptyset$, then extend C_K so that it contains at least one element of S_{irr}. Choose $\tilde{p} \in (C_K \cup \infty) \cap S_{irr}$. From $\Delta(L, \tilde{p})$, we can compute a truncated series for $f^2 = \frac{CA_1A_2^d}{B}$. From it, we can compute a truncated series for f^2B/A_1. Let C be the coefficient of the first term of this sereis, which will done the proof (note that $f^2B/A_1 = CA_2^d$). □

Now the only unknown part of A is A_2. We can assume $A_2 = \sum_{i=0}^{deg(A_2)} b_i x^i$. Since $\deg(A_2) \le \frac{1}{d}d_A \le \frac{1}{3}d_A$, we have

LEMMA 12. *For the rational case, we only need $\frac{1}{3}d_A + 1$ equations to recover A.*

We can not get the equations by the same methods as in Lemma 6 and [6, 7]. If we do so, the equations we get for $\{b_i\}$ will not be linear. The solution to this problem is as follows:

THEOREM 3. *In the rational case, for $A = CA_1A_2^d$, and $A_2 = \sum_{i=0}^{deg(A_2)} b_i x^i$, for each $p \in S_{irr}$ with m_p as pole order of exponent difference, if $p \in C_K$, we will get $\lceil m_p \rceil$ linear equations of $\{b_i\}$.*

PROOF. Since the exponent difference at p will give a $\lceil m_p \rceil$-term truncated series of $g = \frac{A}{B}$ at $x = p$, we can also write B and CA_1 as a series at p. Then we can get the $\lceil m_p \rceil$-term truncated series of $A_2^d = \frac{gB}{CA_1}$. We assume the series is $\sum_{m_p < i \le 2m_p} c_i t_p^{-i}$ where t_p is the local parameter at p. We can rewrite the series as $c_{2m_p} t_p^{-2m_p} S$, where S is a power series with the initial term 1. Let $S_{1/d}$ be a power series with first term 1 such that $S_{1/d}^d = S$. Write $S_{1/d} = 1 + \Sigma_{i>0} a_i t_p^i$ where $a_1, ..., a_{\lceil m_p \rceil - 1}$ are computed by Hensel lifting. Let $\mu_d = \{r \mid r \in C_K, r^d = 1\}$. By Lemma 11 there should be a dth root of c_{2m_p} in C_K. Let c be such a root. Then for each $r \in \mu_d$, let $S_r = ct_p^{-2m_p/d} r S_{1/d}$. Then S_r is a truncated series at p whose dth power is the truncated series of $\frac{gB}{CA_1}$ at p. Then we can also rewrite $A_2 = \sum_{i=0}^{deg(A_2)} b_i x^i$ as a truncated series at p. By comparing the coefficients of S_r and A_2, we will get $\lceil m_p \rceil$ linear equations. Doing this for every $p \in S_{irr}$ provides enough linear equations to find A. Note that we have to try all combinations of $r \in \mu_d$ at every $p \in S_{irr}$. □

REMARK 6. *If $p \notin C_K$, we can use the results from Lemma 7 to get equations. So we can always obtain $\ge \frac{1}{2}d_A$ linear equations, while $\lfloor \frac{1}{3}d_A \rfloor + 1$ equations are sufficient. So we always get enough linear equations.*

REMARK 7. *If we get a candidate (f, d), then $\{f\} \times \{\frac{a}{d} \mid \gcd(a,d) = 1, 1 \le a < \frac{1}{2}d\}$ is a list of candidates for (f, ν).*

To sum up, for all different cases, we have:

THEOREM 4. *From $\Delta(L,p)$, we can always get a list of candidates for (f, ν).*

PROOF. We always have at least $\#S_{reg} + \frac{1}{2}d_A$ linear equations for the coefficients of A. But we may have enough equations (easy case), or only need either 1 (logarithmic case and irrational case) or $\frac{1}{3}d_A + 1$ equations (rational case) to get g. By Remark 4, Remark 5 Remark 7 and Lemma 3, we can also get a finite list of ν. □

The theorem means that we can always find the change of variables. After that, we can compute the projective equivalence to complete the algorithm.

EXAMPLE 3. *Continue with Example 1. We know* $S_{reg} = \{0\}$, $S_{irr} = \{2\}$ *with the truncated series of* g *is* $\Delta = 6t_2^{-4} + 21t_2^{-3} + O(t_2^{-2})$, $B = (x-2)^4$ *and* $d_A = 4$. *Lemma 6 did not provide sufficiently many equations. But for this case the only possible situation is* $A = CxA_2^3$, *and* $A_2 = a_0 + a_1 x$. *The truncated series at* $x = 2$ *of* CA_2^3 *is the series of* $\Delta\cdot(x-2)^4/x$ *at* 2, *which is* $3 + 9t_2 + O(t_2^2)$. *So we can let* $C = 3$. *Then series of* $\frac{gB}{CA_1}$ *is* $S = 1 + 3t_2$. *Since* $K = \mathbb{Q}(x)$, *the only 3rd root of* 1 *is* 1. *So the only possible truncated series which is 3rd root of* S *is* $1 + t_2 + O(t_2^2)$. *And comparing it with* $a_0 + a_1 x = a_0 + 2a_1 + a_1 t_2$, *we get two linear equations* $a_0 + 2a_1 = 1$ *and* $a_1 = 1$. *Solve them we get* $a_0 = -1, a_1 = 1$. *So* $g = \frac{3x(x-1)^3}{(x-2)^4}$.

4. THE ALGORITHM

The input of the algorithm is a differential operator L of order 2. We want to find whether there exists solutions can be represented in terms of bessel functions. If they exist, then find the solutions. Otherwise the algorithm outputs $\emptyset$. Algorithm 1 gives the sketch.

Now we will explain the detail how to retrieve f, ν in different cases.

4.1 Easy Case

In this case, we have enough linear equations from Lemma 6 to recover g. After that, we can use Lemma 3 to get ν. See Algorithm 2 for detail.

4.2 Logarithmic Case

By Remark 4, we can let $\nu = 0$. By Lemma 8, we know all the zeroes of g. We do not yet know the leading coefficient and the multiplicity of each zero. So we can try all combinations of possible multiplicities. Algorithm 3 will give the sketch.

4.3 Irrational Case

In this case, by Lemma 8 we have all the zeroes with multiplicities of g. The only unknown part should be the leading coefficient. But we have at least one linear equations. Algorithm 4 gives the sketch.

4.4 Rational Case

This is the most complicated case. Let $d = \text{denom}(\nu)$ and $f^2 = g = \frac{CA_1A_2^d}{B}$. Algorithm 5 gives the sketch.

5. EXAMPLES

This section will illustrate the algorithm with a few examples[5].

EXAMPLE 4. *Let* $L = \partial^2 + 2 - 10x + 4x^2 - 4x^4$. $K = \mathbb{Q}(x)$
Step 1: We get $S_{reg} = \emptyset$. $S_{irr} = \{\infty\}$ *with the truncated series of* g *at* $x = \infty$ *is* $\frac{4}{9}t_\infty^{-6} - \frac{4}{3}t_\infty^{-4} + O(t_\infty^{-3})$. *So* $d_A = 6$ *and B=1.*
Step 2: It is the rational case with $S_{reg} = \emptyset$. *So* $d \in \{3, 6\}$ *and we can write* $A = CA_2^d$. *If* $d = 3$ *then* $A = CA_2^3$, $A_2 = a_0 + a_1 x + a_2 x^2$. *Since* $B = 1$, *then the truncated*

[5]More examples are given at http://www.math.fsu.edu/∼qyuan

Input: an irreducible differential operator L
Output: solutions represented in terms of Bessel functions if they exist
Find all singularities by factoring the leading coefficient of L over C_K;
foreach *Singularity* p **do**
  compute the generalized exponents at p, then compute the exponent differences and then the truncated series of g
end
Get S_{reg} and S_{irr} according to the generalized exponent differences;
Compute B, d_A (Lemma 4 and 5) and the number of linear equations N ($N \geq \#S_{reg} + \frac{1}{2}d_A$) ;
if $N > d_A$ **then**
  go to **easy case**
else if L *logarithmic at some* $p \in S_{reg}$ **then**
  go to **logarithmic case**
else if *there is* $p \in S_{reg}$ *with* $\Delta(L,p) \notin \mathbb{Q}$ *(i.e* $\nu \notin \mathbb{Q}$*)* **then**
  go to **irrational case**
else
  go to **rational case**
end
`/* It will give us a list of candidates for` (f,ν)`, where` f `is the function of the change of variables, and` ν `is the parameter of Bessel functions */`
foreach (f,ν) *in list of candidates* **do**
  Compute an operator $M_{(f,\nu)}$ such that $L_B \xrightarrow{f}_C M_{(f,\nu)}$;
  Use algorithm described in [1] to compute whether $M_{(f,\nu)} \longrightarrow_{EG} L$ and compute the transformation;
  if *such transformation exists* **then**
    Add the solution to Solutions List
  end
end
Output the solutions list;

Algorithm 1: Main Algorithms

Input: S_{reg}, S_{irr} with truncated series, B, d_A
Output: potential list of (f,ν)
Find all linear equations described in Lemma 6;
Solve linear equations to find f;
if *there is no solution* **then**
  output $\emptyset$
else
  Use Lemma 3 to get a list $\mathcal{N}$ of candidate ν's
end
foreach $\nu \in \mathcal{N}$ **do**
  Add (f,ν) to output list
end

Algorithm 2: Easy Case

Input: S_{reg}, S_{irr} with truncated series, B, d_A
Output: list of (f, ν)
if *not every singularity* $p \in S_{reg}$ *is logarithmic* **then**
  output $\emptyset$
else
  Let $\nu = 0$, $A = a\Pi_{p\in S_{reg}}(x-p)^{a_p}$;
  foreach $\{a_p\}$ *such that* $\Sigma_{p\in S_{reg}} a_p = d_A$ **do**
    Use linear equations described in Lemma 6 to solve a ;
    if *the solution exists* **then**
      Add $(\frac{A}{B}, 0)$ to output list
    end
  end
end

Algorithm 3: Logarithmic case

Input: S_{reg}, S_{irr} with truncated series, B, d_A
Output: list of (f, ν)
Use Lemma 8 find all zeroes and multiplicities;
Use linear equations given by Lemma 6 to get the leading coefficient;
Use Lemma 3 to get a list of candidates for ν's;
Add solutions to output list;

Algorithm 4: Irrational case

Input: S_{reg}, S_{irr} with truncated series, B, d_A
Output: list of (f, ν)
if $S_{reg} = \emptyset$ **then**
  Let the list of candidates for d be the set of factors of d_A;
  Let $A_1 = 1$;
else
  Use Lemma 10 to get a list of candidates for d and A_1
end
foreach *candidate* (d, A_1) **do**
  Fix C by Lemma 11;
  Use linear equations given by Theorem 3 to compute A_2 ;
  If a solution exists, add $\{f\} \times \{\frac{a}{d} \mid \gcd(a,d) = 1, 1 \le a < \frac{1}{2}d\}$ to output list
end

Algorithm 5: Rational case

series of gB is the same as g. So we can let $C = \frac{4}{9}$. Then the truncated series of A_2^3 is $t_\infty^{-6} + 3t_\infty^{-4} = t_\infty^{-6}(1-3t_\infty^2)$. Since the only 3rd root of 1 in C_K is 1, then the only 3rd root of $1-3t_\infty^2$ is $1-t_\infty^2$. So by comparing coefficients of $t_\infty^{-2}(1-t_\infty^2)$ and $A_2 = a_0 + a_1 t_\infty^{-1} + a_2 t_\infty^{-2}$, we can get $A_2 = x^2 - 1$ and then $g = \frac{4}{9}(x^2-1)^3$. We can do this process for $d = 6$, in this case, we have no solution. So we have $(\frac{2}{3}\sqrt{x^2-1})^3, \frac{1}{3})$ as the only possible candidate.

Step 3: We compute $L_B \xrightarrow{f}_C M$, and then the projective equivalence from M to L. Combining these transformations produces the following solutions of L:

$$C_1\Big(\frac{2(2x^4+x^3-3x^2+x+2)}{\sqrt{x^2-1}} I_{\frac{1}{3}}\big(\frac{2}{3}\sqrt{(x^2-1)^3}\big)$$
$$+ 2(2x+1)(x^2-1) I_{\frac{4}{3}}\big(\frac{2}{3}\sqrt{(x^2-1)^3}\big)\Big)$$
$$+ C_2\Big(\frac{2(2x^4+x^3-3x^2+x+2)}{\sqrt{x^2-1}} K_{\frac{1}{3}}\big(\frac{2}{3}\sqrt{(x^2-1)^3}\big)$$
$$- 2(2x+1)(x^2-1) K_{\frac{4}{3}}\big(\frac{2}{3}\sqrt{(x^2-1)^3}\big)\Big)$$

Example 5. *Consider the operator:*

$$L := \partial^2 - \frac{15x^4-30x^3+x^2+8x-4}{x(x-1)(15x^3-10x^2+9x-4)}\partial - \frac{1}{36x^2(15x^3-10x^2+9x-4)(x-1)^2}(30375x^{20} - 212625x^{19} + 733050x^{18} - 170595x^{17} + 3034305x^{16} - 435055x^{15} + 5166936x^{14} - 5172228x^{13} + 4401369x^{12} - 3189159x^{11} + 1962738x^{10} - 1016622x^9 + 434943x^8 - 149229x^7 + 38844x^6 - 3933x^5 - 4554x^4 + 3789x^3 - 1612x^2 + 432x - 64).$$

Step 1: The singularity are ∞, RootOf$(15_Z^3 - 10_Z^2 + 9_Z - 4), 1, 0$.

Step 2: $S_{reg} = \{1, 0\}$, with the exponent difference $\frac{5}{3}$ and $\frac{4}{3}$ respectively. We also have $S_{irr} = \{\infty\}$ and the truncated series of g at $x = \infty$ is $t_\infty^{-15} - 5t_\infty^{-14} + 13t_\infty^{-13} - 25t_\infty^{-12} + 38t_\infty^{-11} - 46t_\infty^{-10} + 46t_\infty^{-9} - 38t_\infty^{-8} + O(t_\infty^{-7})$. So $B = 1$ and $d_A = 15$.

Step 3: we can easily verify that this is a rational case. Since the exponent difference of at 0 and 1 both have denominator 3, so d is a multiple of 3. If $d = 3$ then $A = Cx^2(x-1)A_2^d$ or $A = Cx(x-1)^2A_2^d$. If $d = 6$, then the multiplicity of both 1 and 0 should be a multiple of $\frac{6}{3} = 2$ then it will contradict with $\deg(A) = 15$. Similarly $A = Cx^3(x-1)^3A_2^9$, $A = Cx^5(x-1)^{10}$ and $A = Cx^{10}(x-1)^5$ are candidates as well. Then we compute each candidate by the method in Theorem 3. Finally, we get $f = \sqrt{x^4(x-1)^5(x^2+1)^3}$ and $\nu = \frac{1}{3}$ is the only remaining candidate .

Step 4:Let $L_B \xrightarrow{f}_C M$. Now M is already equal to L. So the general solution is $C_1 I_{\frac{1}{3}}(\sqrt{x^4(x-1)^5(x^2+1)^3}) + C_2 K_{\frac{1}{3}}(\sqrt{x^4(x-1)^5(x^2+1)^3})$

6. CONCLUSION

In this paper, we developed an algorithm to solve second order differential equations in terms of Bessel functions. We extended the algorithm described in [7] which already solved the problem in the $f \in \mathbb{C}(x)$ case, but not in the square root case. We implemented the algorithm in Maple. The code and examples can be downloaded from http://www.math.fsu.edu/∼qyuan. A future task is to try to develop a similar algorithm to find ${}_2F_1$ type solutions.

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