Solving Third Order Linear Differential Equations in Terms of Second Order Equations

Mark van Hoeij (Florida State University)

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Talk presented by:

George Labahn (University of Waterloo)

Notations.

Notation: $\partial = \frac{\mathrm{d}}{\mathrm{d}x}$. Differential operator:

$$L = \partial^3 + c_2\partial^2 + c_1\partial + c_0$$

where the $c_i = c_i(x)$ are rational functions.

$$L(y) = y''' + c_2 y'' + c_1 y' + c_0 y.$$

Solution space of L:

$$V(L) = \{y \mid L(y) = 0\}.$$

Dimension $=$ order $L = 3$.

Goal: Find closed form expressions for solutions of $L(y) = 0$, by trying to reduce this equation to equation(s) of lower order.

Singer's Theorem

Suppose L has rational function coefficients and **order 3**.

Suppose that L can be solved in terms of solutions of lower order equations (*again linear with rational functions as coefficients*).

Then [Singer 1985] at least one of the following is true:

1. L is **reducible** (L can be factored).
2. L is a **symmetric square**.
3. L is **gauge equivalent** to a symmetric square.

(Note: $\exists$ generalizations to higher order by M. Singer, A. Person, K. Nguyen, M. van der Put)

Goal: reduce order whenever possible.

Cases 1 and 2 are already handled by existing implementations.

Goal: Handle case 3 as well.

This way, reduction of order 3 to lower order will be done whenever it is possible.

Reduction of Order, Case 1 (the easiest case): Is L the symmetric square of a second order operator?

Let $L = \partial^3 + c_2\partial^2 + c_1\partial + c_0$.

L is a **symmetric square** if there exist functions y_1, y_2 such that $y_1^2,\ y_1y_2,\ y_2^2$ is a basis of $V(L)$.

In this case y_1, y_2 are solutions of some operator:

$$L_2 = \partial^2 + a_1\partial + a_0.$$

Testing if L is a symmetric square (and finding L_2) is easy:

Take

$$a_1 := \frac{c_2}{3}, \qquad a_0 := \frac{c_1 - a_1' - 2a_1^2}{4}$$

and test if $4a_0a_1 + 2a_0'$ equals c_0. (19'th century).

Reduction of Order, Case 2: Factorization.

Let $L = \partial^3 + c_2\partial^2 + c_1\partial + c_0$.

L is **reducible** if there exist operators $L_1, L_2 \in \mathbb{C}(x)[\partial]$ of order less than 3 such that

$$L = L_1 \cdot L_2$$

where multiplication = composition of operators.

If L is reducible then we can reduce the order, and there are algorithms + implementations to do this.

Remains to do: a practical way to handle Case 3.

Reduction of Order, Case 3.

Let $L \in \mathbb{C}(x)[\partial]$ have order 3. If $L(y) = 0$ can be reduced to lower order, then according to [Singer 1985] one of three cases must hold. Two out of those three cases are already implemented.

Thus, if a reduction of order is possible, and currently available software does not already perform this reduction then we must be in Case 3. Then:

L is **gauge equivalent** to the **symmetric square** of a **second order operator**.

(Definitions will come later, first an example.)

Example.

$$L = \partial^3 + \frac{x-2}{(x-1)x}\partial^2 - \frac{4x^2+3x+2}{9(x-1)x^2}\partial + \frac{2}{9(x-1)x}$$

This example was constructed in such a way that it has solutions that can be expressed in terms of Bessel functions.

Hence: a reduction to order 2 is possible.

However, currently available software does not find a reduction of order, so we must be in Case 3 by Singer's theorem.

Example, continued.

The example L is **irreducible** and **not a symmetric square**. But it is solvable in terms of lower order equations.

Hence L must be gauge equivalent to the symmetric square of some second order L_2 by [Singer 1985].

To solve L we want to find such L_2 and then solve L_2.

Find such L_2:

- Solved in theory in [Singer 1985], but this algorithm would be too slow for almost all examples; it involves solving large systems of polynomial equations (e.g. with Gröbner basis).
- **Our contribution in ISSAC'2007:** An efficient algorithm (does not require solving systems of non-linear equations).

Gauge transformations.

Two operators $L, \tilde{L}$ of the same order are called **gauge equivalent** if there exists an operator G that maps the solutions of L onto the solutions of $\tilde{L}$.

Given $L, \tilde{L}$ such **gauge transformation** G can be found (if it exists) using the **Homomorphisms** command in Maple 10.

Let L and G be linear differential operators.
$V(L) =$ solution space of L.
Apply operator G to $V(L)$. The result $G(\,V(L)\,)$ is the solution space of an operator we denote as L^G.

If L is gauge equivalent to $\tilde{L}$ then $\tilde{L} = L^G$ for some G.

Gauge transformations, an example.

Take

$$L_a \;=\; \partial^2 - x \qquad\qquad L_b \;=\; \partial^2 - \frac{1}{x}\partial - x.$$

L_a corresponds to the Airy equation $y'' - xy = 0$.

L_a and L_b are gauge equivalent because $L_b = L_a^G$ where

$$G = \partial \qquad (= \text{ differentiation}).$$

This means that by applying G to the solutions of L_a we get the solutions of L_b.

The inverse of this gauge transformation is $\frac{1}{x}\partial$. It sends solutions of L_b back to solutions of L_a.

Notation and Problem Statement.

Denote $\mathrm{Sym}^2(L)$ as the **symmetric square** of L.
Then y^2 is a solution of $\mathrm{Sym}^2(L)$ for every $y \in V(L)$.

To solve L from the example we need to find a gauge transformation that turns L into a symmetric square.

Thus, we search for $G \in \mathbb{C}(x)[\partial]$ for which there exists L_2 with

$$L^G = \mathrm{Sym}^2(L_2).$$

Chicken and Egg Problem.

Input: L.

To find: (if it exists) a gauge transformation $G \in \mathbb{C}(x)[\partial]$ such that

$$L^G = \mathrm{Sym}^2(L_2)$$

for some L_2.

- If we had G, then it's easy to compute L^G, and then it's trivial to compute L_2.
- If we had L_2, then we can compute $\mathrm{Sym}^2(L_2)$ to get L^G, from which we can compute G with Maple 10.

However, we have neither L_2 nor G, we only have L.

How to find a gauge transformation G that takes L to the symmetric square of some L_2 if we don't know L_2?

Goal: Find some $G = g_2\partial^2 + g_1\partial + g_0$ for which L^G is a symmetric square.

This means L^G has solutions z_1, z_2, z_3 of the form

$$z_1 = y_1^2, \quad z_2 = y_1 y_2, \quad z_3 = y_2^2$$

for some functions y_1, y_2. Then

$$z_1 z_3 - z_2^2 = 0.$$

This suggests that there should be some quadratic relation for g_0, g_1, g_2 (the solutions z_i of L^G depend linearly on g_0, g_1, g_2).

How to find G?

Indeed, according to [Singer 1985], if L can be solved in terms of second order equations, then in the non-trivial case, the operator $\mathrm{Sym}^2(L)$ has a first order factor $\partial - r$, and there exist some quadratic relation R such that:

L^G is a symmetric square $\iff$ $R(g_0, g_1, g_2) = 0$
(here $G = g_2\partial^2 + g_1\partial + g_0$)

To solve $L(y) = 0$, we need:

1. A formula for R in terms of L and r.
2. An algorithm to solve quadratic relations $R = 0$ over $\mathbb{C}(x)$.
3. Compute L^G and write it as $\mathrm{Sym}^2(L_2)$ for some L_2.
4. Solve L_2.

Our contribution.

The formula for R in [Singer 1985] is not very explicit and contains two unknown constants which makes solving $R = 0$ very costly.

We give a more explicit formula for R that is less work to calculate, and which moreover contains no unknown constants.

This way $R = 0$ can be solved efficiently, and the algorithm becomes practical.

(How to find this formula will be discussed later, first an example.)

Example.

$$L = \partial^3 + \frac{x-2}{(x-1)x}\partial^2 - \frac{4x^2+3x+2}{9(x-1)x^2}\partial + \frac{2}{9(x-1)x}.$$

$\mathrm{Sym}^2(L)$ has order 6, and has $\partial - 1/(x+\frac{1}{4})$ as a factor. Substituting this into the formula for R from [ISSAC'2007] gives:

$$R(g_0, g_1, g_2) = 81\,x^3\,(1+4\,x)^4\,{g_0}^2 + 72\,x^2\,(x-2)\,(1+4\,x)^2\,g_0\,g_2 - 36\,x^2\,(x-2)\,(1+4\,x)^2\,{g_1}^2 + 72\,x\,(1+4\,x)\left(4\,x^2-12\,x-1\right)g_1\,g_2 + \left(256\,x^4-368\,x^3+2544\,x^2+484\,x+16\right){g_2}^2 \;=\; 0.$$

Such quadratic relations can be solved with [Schicho, ISSAC'1998] or [vH + Cremona, 2006] (implementations in Maple and Magma are available for download).

Example, continued.

Solving $R = 0$ requires making arbitrary choices, so each time you run it you may get a different solution (g_0, g_1, g_2).

Each solution g_0, g_1, g_2 gives an operator

$$G = g_2\partial^2 + g_1\partial + g_0.$$

One such solution gives

$$G \;=\; 9\partial^2 + \frac{3(x+2)}{x}\partial - \frac{2}{x}.$$

Then one can compute

$$L^G \;=\; \partial^3 + \frac{3(x-2)}{x(x-1)}\partial^2 - \frac{2\left(2\,x^3-8\,x^2+10\,x-31\right)}{9x^2(x-1)^2}\partial - \frac{2(x^4-5\,x^3+11\,x^2+16\,x+4)}{9x^3(x-1)^3}$$

which is a symmetric square.

Example, continued.

We can write L^G as the symmetric square of

$$L_2 = \partial^2 + \frac{(x-2)}{x\,(x-1)}\partial - \frac{4\,x^3 - 7\,x^2 - 16\,x - 8}{36x^2\,(x-1)^2}.$$

Every solution y of L_2 gives a solution y^2 of L^G, and by applying the inverse gauge transformation we get a solution of L.

Take for example

$$y = \frac{\sqrt{x-1}\;B_I\left(\frac{1}{3}, \frac{2}{3}\sqrt{x}\right)}{\sqrt{x}}$$

where B_I is the Bessel I function.

Example, continued.

Now y^2 is a solution of L^G. The inverse of gauge transformation G can be represented as

$$\frac{9x^3}{x-1}\partial^2 + \frac{3x^2\,(5\,x-11)}{(x-1)^2}\partial - \frac{x\,(4\,x^3-9\,x^2-16+3\,x)}{(x-1)^3}.$$

Applying this operator to y^2 we find the following solution of L

$$\sqrt{x}\,b_1\,b_2 - xb_1^2 + xb_2^2$$

where

$$b_1 = B_I\left(\frac{1}{3}, \frac{2}{3}\sqrt{x}\right) \quad \text{and} \quad b_2 = B_I\left(-\frac{2}{3}, \frac{2}{3}\sqrt{x}\right).$$

How to find a formula for R.

The gauge transformation G maps the solution space of L to the solution space of L^G

$$G : V(L) \to V(L^G).$$

This induces a map:

$$G_2 : V(\mathrm{Sym}^2(L)) \to V(\mathrm{Sym}^2(L^G)).$$

Goal: $L^G = \mathrm{Sym}^2(L_2)$ for some L_2. Then $\mathrm{Sym}^2(L^G) = \mathrm{Sym}^4(L_2)$ has order 5. In the non-trivial case (if L is not a symmetric square) then $\mathrm{Sym}^2(L)$ has order 6. So G_2 has a 1-dimensional kernel. Then

$$G_2(Y) = 0$$

where Y is an exponential solution of $\mathrm{Sym}^2(L)$.

How to find a formula for R.

The coefficients of G_2 depend quadratically on the coefficients g_0, g_1, g_2 of G. A general formula for G_2 expressed in terms the coefficients of L and G can be computed with a computer algebra system.

This general G_2 can then be inserted into the implementation.

Finding the quadratic relation $R(g_0, g_1, g_2) = 0$ then becomes easy to implement: Just substitute the coefficients of L into $G_2(Y)$ where Y is as in the previous slide.

(This already leads to a practical implementation. In the paper additional effort is made to make the formula as small as possible, so that it can easily be typed into future implementations.)

Current Status.

The reduction of order 3 to order 2 is implemented, and the code is available. One issue remains:

Solving quadratic relation $R(g_0, g_1, g_2) = 0$
$\Longrightarrow$ solution is not unique
$\Longrightarrow$ gauge transformation $G = g_2\partial^2 + g_1\partial + g_0$ not unique
$\Longrightarrow$ L^G is not unique
$\Longrightarrow$ L_2 is not unique

So for the same third order L one can end up with different L_2's. If one of them is solvable in closed form, then so is the other.

However, it frequently happens that current computer algebra systems solve one of those L_2's but not the other.

What comes next.

So at the moment, whether or not we find closed form solutions for L depends on which L_2 our program happened to find, which depends on chance. To fix this, the next research goal will be:

Next Goal: (will be supported by NSF 0728853): Solve any second order operator $L_2 \in \mathbb{C}(x)[\partial]$ that has closed form solutions.

Thank you for your attention.