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SOLUTIONS OF SECOND ORDER RECURRENCE RELATIONS

By

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To my parents, Henry and Ann Levy

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ABSTRACT

This thesis presents three algorithms each of which returns a transformation from a base equation to the input using transformations that preserve order and homogeneity (referred to as gt-transformations). The first and third algorithm are new and the second algorithm is an improvement over prior algorithms for the second order case.

The first algorithm 'Find ${}_2F_1$' finds a gt-transformation to a recurrence relation satisfied by a hypergeometric series $u(n) = {}_2F_1\left(\begin{matrix} a+n & b \\ & c \end{matrix}\middle| z\right)$, if such a transformation exists.

The second algorithm 'Find Liouvillian' finds a gt-transformation to a recurrence relation of the form $u(n+2) + b(n)u(n) = 0$ for some $b(n) \in \mathbb{C}(n)$, if such a transformation exists.

The third algorithm 'Database Solver' takes advantage of a large database of sequences, 'The On-Line Encyclopedia of Integer Sequences' maintained by Neil A. J. Sloane at AT&T Labs Research. It employs this database by using the recurrence relations that they satisfy as base equations from which to return a gt-transformation, if such a transformation exists.

CHAPTER 1

INTRODUCTION

There exist many algorithms, implemented in Computer Algebra Systems (CAS), for solving second order recurrence relations in terms of first order relations, i.e. hypergeometric terms. Unfortunately, when no such solution exists the CAS user is frequently left without any information. This thesis presents three algorithms that provide more information in that event. Common to each algorithm is a transformation from a base equation to the input using transformations that preserve order and homogeneity (referred to as gt-transformations).

The third algorithm and the examples below employ 'The On-Line Encyclopedia of Integer Sequences' or OEIS ([1]) maintained by Neil A. J. Sloane at AT&T Labs Research. As its name implies, the OEIS contains a great deal of information about many sequences beyond just the terms of the sequence. Some information that may be found for a particular sequence is description, formulas, references (e.g. to papers, books, and websites), cross references to other sequences, CAS code, examples, and the author of that entry.

The first algorithm called by our main program is 'Find ${}_2F_1$' from Chapter 4. This algorithm finds a gt-transformation to a recurrence relation satisfied by a hypergeometric series $u(n) = {}_2F_1\left(\begin{matrix} a+n & b \\ c & \end{matrix}\middle| z\right)$, if such a transformation exists. For an example, sequence $A005572 = [1, 4, 17, 76, 354, 1704, 8421, \dots]$ from the OEIS represents the "Number of walks on cubic lattice starting and finishing on the xy plane and never going below it." $A005572$ has offset 0 (i.e. the first entry in the list is $A005572(0)$) and satisfies:

$$(12n + 12)A005572(n) + (-20 - 8n)A005572(n + 1) + (n + 4)A005572(n + 2) = 0$$

We don't get any result from Maple's recurrence equation solver, 'rsolve,' but the output from our program is:

$$A005572(n) = -\_c2^n \frac{-2\text{hypergeom}([\frac{1}{2}, n + 2], [1], \frac{2}{3}) + 3\text{hypergeom}([\frac{1}{2}, n + 1], [1], \frac{2}{3})}{n + 2}$$

The second algorithm is 'Find Liouvillian' from Chapter 5. This algorithm finds a gt-transformation to a recurrence relation of the form $u(n+2) + b(n)u(n) = 0$ for some $b(n) \in \mathbb{C}(n)$, if such a transformation exists. 'Find Liouvillian' is not unique in terms of its purpose but, for second order recurrence relations, it is faster than prior algorithms. We note here, so that the name will not be misleading, that if such a transformation exists then both the input and the output are Liouvillian. This algorithm can be useful because $u(n+2) + b(n)u(n) = 0$ is easily solved with (interlaced) hypergeometric terms, e.g.

$$2u(n+2) - (n+3)u(n) = 0 \text{ has solutions: } \begin{cases} k_1\,\Gamma\left(\dfrac{n+3}{2}\right), & \text{if } n \text{ even} \\ k_2\,\Gamma\left(\dfrac{n+3}{2}\right), & \text{if } n \text{ odd} \end{cases} \text{ with } k_1, k_2 \in \mathbb{C}$$

An example of this algorithm uses $A099364$ from the OEIS ($A099364$ is "An inverse Chebyshev transform of $(1-x)^2$" and has offset 0). When we input the recurrence relation satisfied by $A099364$ into 'rsolve' we don't get a result, so we enter it into our program and get the solution:

$$A099364(n) = \left(\frac{1}{6}n + \frac{5}{6}\right)v(n) - \left(\frac{1}{12}n + \frac{1}{2}\right)v(n+1), \quad v(n+2) - \frac{4(n+2)}{n+7}v(n) = 0$$

The third algorithm is 'Database Solver' from Chapter 6. This algorithm takes advantage of a large database of sequences, 'The On-Line Encyclopedia of Integer Sequences' or OEIS ([1]), by using the recurrence relations that they satisfy as base equations. In addition to these sequences we use sequences that satisfy third order equations that are the Least Common Left Multiple of a second order and a first order relation. In order that a search does not take too much time, we have already searched the database and generated collections such that there exists a gt-transformation between any two members of a collection. It would still take much time to search for transformations from representatives from each group so we first check certain invariants. For each collection we choose two representatives assuming that, after an appropriate gt-transformation, we have two linearly independent sequences in S (see Definition 2.6). If the input is a recurrence relation with initial conditions defining a sequence then the output will be a transformation from one or two representative sequences (if there are matches in our database). The output could be useful since the representatives are chosen, in large part, for how much information there is about that sequence (formulas, papers citing the sequence, . . .).

As an example of this algorithm, suppose we were working on "Coefficients of series whose square is the weight enumerator of the [8,4,4] Hamming code" (this is actually sequence $A108095$

from the OEIS). This sequence, $[1, 7, -24, 168, -1464, 14280, \dots]$ has offset 0 and satisfies:

$$(n-1)u(n) + (7+14n)u(n+1) + (n+2)u(n+2) = 0$$

Again, 'rsolve' does not return a result so we enter the following into our program:

$$\text{findrel}((n-1)u(n) + (7+14n)u(n+1) + (n+2)u(n+2), u(n), u(0) = 1, u(1) = 7);$$

and we get the following output:

$$u(n) = 7(-1)^n \left(\frac{(97n+49)A084768(n)}{7n(n-1)} - \frac{(n+1)A084768(n+1)}{n(n-1)} \right)$$

There is a little more information on $A084768$'s page then there is on the page for $A108095$, but we also see that it is related to "$P_n(7)$, where P_n is n'th Legendre polynomial" which is interesting on its own.

An implementation of the algorithms contained in this thesis, along with a Maple demonstration worksheet, is available at [11].

CHAPTER 2

PRELIMINARIES

Definition 2.1. *τ will refer to the shift operator acting on $\mathbb{C}(n)$ and $Mat_{a\times b}(\mathbb{C}(n))$ by $\tau\colon n \mapsto n+1$.*

An operator $L = \sum_i a_i\tau^i$ acts as $Lu(n) = \sum_i a_i u(n+i)$.

Definition 2.2. *$\mathbb{C}(n)[\tau]$ is the ring of* linear difference operators *where ring multiplication is composition of operators $L_1L_2 = L_1 \circ L_2$, e.g. $(\tau - a(n))(\tau - b(n)) = \tau^2 - (a(n) + b(n+1))\tau + a(n)b(n)$.*

Definition 2.3. *A* linear recurrence relation *is an equation defining a term of a sequence as a linear combination of previous terms, i.e. $u(n+k) = a_{k-1}(n)u(n+k-1) + \cdots + a_0(n)u(n),\ a_i(n) \in \mathbb{C}(n)$.*

We let τ operate on $u(n) \in \mathbb{C}^{\mathbb{N}}$ by $u(n) \mapsto u(n+1)$. We consider homogeneous difference equations with non-constant coefficients of the form: $Lu = 0,\ L \in \mathbb{C}(n)[\tau],\ u \in \mathbb{C}^{\mathbb{N}}$, i.e. $u\colon \mathbb{N} \to \mathbb{C}$. (Where $\mathbb{N} = \{0, 1, 2, \dots\}$.) e.g. for $L = \tau^2 + a(n)\tau + b(n)$ with $a(n), b(n) \in \mathbb{C}(n)$ the equation $Lu = 0$ reads:

$$u(n+2) + a(n)u(n+1) + b(n)u(n) = 0$$

Our goal is to solve such difference equations in terms of well-known functions or sequences.

Definition 2.4. *We define a* sequence *as a function with domain the non-negative integers (possibly minus finitely many integers). It is in this sense that we will refer to a function as a sequence.*

Definition 2.5. *The* offset *of a sequence will refer to the smallest integer in its domain. If not specified then* offset *will be assumed to be* 0. *For example, the sequence* $[0, 1, \dots]$ *refers to* $u(0) = 0, u(1) = 1, \dots$ *if the offset* $= 0$*, but it refers to* $u(1) = 0, u(2) = 1, \dots$ *if the offset* $= 1$.

Preferred offset varies in the literature somewhat. The convention of the On-Line Encyclopedia of Integer Sequences (OEIS) is offset 1 while Maple (specifically its recurrence equation solver ‘rsolve’) mostly uses offset 0. Let $L \in \mathbb{C}(n)[\tau]$, $u \in \mathbb{C}^{\mathbb{N}}$ such that $Lu = 0$. If $L \notin \mathbb{C}[\tau]$ then L and u are dependent on offset.

A change in offset of $k \in \mathbb{Z}$ given by $v(n) = u(n+k)$ is a simple example of a gauge transformation, one of the transformations we will later introduce (see Definition 2.15).

Definition 2.6 (Singer)**.** *Let equivalence classes of sequences be defined by $S = \mathbb{C}^{\mathbb{N}}/\sim$ where $s_1 \sim s_2$ if there exists $N \in \mathbb{N}$ such that, for all $n > N$, $s_1(n) = s_2(n)$.*

The reason for using S is that the dimension of the solution space will be equal to the order of the recurrence operator (see Theorem 2.1 below).

Remark 2.1. *Working in S also enables us to work in $\mathbb{C}[n][\tau]$ as well as in $\mathbb{C}(n)[\tau]$. In particular, if $L \in \mathbb{C}(n)[\tau]$ and we multiply away the denominators of the coefficients to obtain an element of $\mathbb{C}[n][\tau]$ then the solution space does not change when working in S.*

Definition 2.7. *A* unit *is a sequence in S that is invertible, i.e. a sequence that only has finitely many zeros.*

Theorem 2.1. *[‘A=B’ Theorem 8.2.1] Let $L = \sum_{k=0}^{r} a_k \tau^k$ be a linear recurrence operator of order r on S. If a_r and a_0 are units, then* $\dim(\ker(L)) = r$.

We can view $\mathbb{C}(n)$ as a subset of S so the theorem applies to $L \in \mathbb{C}(n)[\tau]$ with $a_0, a_r \neq 0$.

Definition 2.8. *A function or sequence $v(n)$ such that $v(n+1)/v(n) = r(n)$ is a rational function of n will be called a* hypergeometric term*. Such a $v(n)$ will be called a* hypergeometric solution *of any $L \in \mathbb{C}(n)[\tau]$ for which $Lv = 0$ and will correspond to a first order right hand factor of L, namely $\tau - r(n) = \tau - v(n+1)/v(n)$.*

In the literature, *hypergeometric function* is usually synonymous with a convergent *hypergeometric series* $\sum_{n=0}^{\infty} v(n)x^n$ where $v(n)$ is a hypergeometric term.

Existing algorithms can solve all second order linear recurrence operators with:

1. first order right hand factors (i.e. operators with hypergeometric solutions).

2. constant coefficients (this will be a subset of (1) when working in an algebraically closed field of constants).

Example 2.1. *The operator* $L = \tau^2 - \tau - 1$ *satisfied by the Fibonacci numbers,* $F(n) = 0, 1, 1, 2, 3, 5, 8, 13, \ldots,$ *is an example (of both* (1) *and* (2) *above) of an operator that is already solved using existing algorithms:*

$$L = \left(\tau - \frac{1}{2} - \frac{\sqrt{5}}{2}\right)\left(\tau - \frac{1}{2} + \frac{\sqrt{5}}{2}\right)$$

$$F(n) = \frac{1}{\sqrt{5}}\left(\left(\frac{1}{2} + \frac{\sqrt{5}}{2}\right)^n - \left(\frac{1}{2} - \frac{\sqrt{5}}{2}\right)^n\right)$$

Definition 2.9. *A* monic rational function *refers to a rational function* $r = f/g$ *such that* f, g *are monic polynomials.*

Definition 2.10. Γ *will represent the usual* gamma function *defined as*

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \tag{2.1}$$

for $Re(z) > 0$ *and, for* $-z \notin \mathbb{N}$*, by its analytic continuation to the complex plane* ($\Gamma(z) = \Gamma(z+n)/(z(z+1)\cdots(z+n-1))$ *is used to continue the function analytically).*

Remark. *The gamma function was defined to be an extension of the factorial function and they are related by* $\Gamma(n) = (n-1)!,\ n \in \mathbb{N}^*$*. This can be seen from the identity* $\Gamma(z+1) = z\,\Gamma(z)$ *which can be obtained (after 'integration by parts') from Equation* (2.1).

Definition 2.11. *The* Pochhammer symbol, $(r)_n$ *with* $n \in \mathbb{N}$*, will be used to represent the* shifted factorial *given by:*

$$(r)_n = r(r+1)\cdots(r+n-1)$$

$$(r)_0 = 1$$

In the literature (but not in this thesis) the shifted factorial is also called the rising factorial *and may be denoted* $r^{(n)}$ *and the Pochhammer symbol is sometimes used to represent the falling factorial.*

Note: For $n \in \mathbb{C}$, $(r)_n$ *can be extended analytically to the complex plane using*

$$(r)_n = \frac{\Gamma(r+n)}{\Gamma(r)}$$

Definition 2.12. *A* holonomic function *is a* $u(n) \in \mathbb{C}^{\mathbb{N}}$ *such that* $Lu = 0$ *for some* $L \in \mathbb{C}[n][\tau]$ *(i.e. L is a linear, homogeneous difference operator having polynomial coefficients).*

Definition 2.13. *$V(L)$ refers to the solution space of the operator L, i.e. $V(L) = \{u \in S \mid Lu = 0\}$, where S is as in Definition 2.6.*

Example 2.2. *For $L = \tau + n + 1$ we write $V(L) = \mathbb{C} \cdot (-1)^n \Gamma(n+1)$ or $V(L) = \mathbb{C} \cdot [1, -1, 2, -6, 24, -120, \dots]$.*

Let $D = \mathbb{C}(n)[\tau]$. If $L \in D$ with $L \neq 0$ then D/DL is a $D-$module.

Definition 2.14. *L_1 is* gauge equivalent *to L_2 when D/DL_1 and D/DL_2 are isomorphic as $D-$modules.*

Lemma 2.1. *L_1 is gauge equivalent to L_2 if and only if there exists $G \in D$ such that $G(V(L_1)) = V(L_2)$ and L_1, L_2 have the same order. Thus G defines a bijection $V(L_1) \to V(L_2)$.*

Note: If $D/DL_1 \cong D/DL_2$ then G in the Lemma corresponds to the image in D/DL_1 of the element $1 \in D/DL_2$.

Definition 2.15. *The bijection defined by G in Lemma 2.1 above will be called a* gauge transformation.

Definition 2.16. *The sequence obtained by multiplication of the corresponding terms of two sequences will be called a* term product *of those sequences (e.g. $w(n) = v(n)u(n)$, $n = 0, 1, \dots$). The term product of a sequence, $u(n)$, with a hypergeometric sequence will be called a* t-transformation *of $u(n)$. The term product of two difference operators, represented as $L_1 \otimes L_2$, for which $v(n), u(n)$ are respective solutions, is defined to be the minimal difference operator satisfied by $w(n) = v(n)u(n)$. A t-transformation as an operator (a hypergeometric sequence times an, as yet, unspecified sequence) will be represented as $(\tau - r)\otimes$ for some $r \in \mathbb{C}(n)$, e.g. a t-transformation operating on L would be represented as $(\tau - r) \otimes L$ whose result is the difference operator satisfied by: $s(n)u(n)$ for $s(n)$ any solution of $\tau - r$ and for each solution, $u(n)$, of L.*

In the above definition, let the factorization of r over $\mathbb{C}$ be given as $c(n - r_1)^{e_1} \cdots (n - r_m)^{e_m}$ with $c, r_i \in \mathbb{C}$ and $e_i \in \mathbb{Z}$. The solution (unique up to multiplication by constants), $s(n)$, of $\tau - r$ is given by

$$s(n) = c^n \Pi_{i=1}^m (\Gamma(n - r_i))^{e_i}$$

Let $L = \tau^2 + a(n)\tau + b(n)$ then

$$(\tau - r(n)) \otimes L = \tau^2 + a(n)r(n+1)\tau + b(n)r(n)r(n+1)$$

Definition 2.17. *A* gt-transformation *will refer to a gauge transformation followed by a t-transformation.*

Example 2.3. *Examples of hypergeometric terms (i.e. solutions of first order operators)*

- *The solution of* $\tau - 2$ *is* 2^n
- *The solution of* $\tau - 2n$ *is* $2^n\Gamma(n)$
- *The solution of* $\tau - 5(n-2)^2(n+1)$ *is* $5^n(\Gamma(n-2))^2\Gamma(n+1)$.

Note: Solutions of first order difference operators are unique up to a constant.

Definition 2.18. *Let* $\hat{r}(n) = cp_1(n)^{e_1} \cdots p_j(n)^{e_j} \in C(n)$ *with* $C \subseteq \mathbb{C}$. *Let the* $e_i \in \mathbb{Z}$*, let the* $p_i(n)$ *be irreducible in* $C[n]$*, and let* $s_i \in C$ *equal the sum of the roots of* $p_i(n)$. $\hat{r}(n)$ *is said to be in* shift normal form *if* $-\deg(p_i(n)) < \mathrm{Re}(s_i) \leqslant 0$*, for* $i = 1, \ldots, j$. *We denote* $\mathrm{SNF}(r(n))$ *as the* shift normalized form *of* $r(n)$ *which is obtained by replacing each* $p_i(n)$ *by* $p_i(n+k_i)$ *for some* $k_i \in \mathbb{Z}$ *such that* $p_i(n+k_i)$ *is in shift normal form. Two rational functions,* $r_1(n), r_2(n)$ *will be called* shift equivalent, $r_1(n) \overset{\mathrm{SE}}{\equiv} r_2(n)$*, if* $\mathrm{SNF}(r_1(n)) = \mathrm{SNF}(r_2(n))$.

Theorem 2.2. *Let* $Lu(n) = 0$ *be a monic linear recurrence equation with* $L \in \mathbb{C}(n)[\tau]$. *There exists a basis of solutions in* $\mathbb{Q}^{\mathbb{N}}/\!\sim$ *if and only if* $L \in \mathbb{Q}(n)[\tau]$

Example 2.4. *A relatively simple example* $u(n+2) + 7u(n+1) - 4u(n) = 0$ *with initial conditions* $\{u(0) = 1,\ u(1) = 1\}$ *has solution*

$$u(n) = \left(-\frac{9\sqrt{65}}{130} + \frac{1}{2}\right)\left(-\frac{7}{2} - \frac{\sqrt{65}}{2}\right)^n + \left(\frac{9\sqrt{65}}{130} + \frac{1}{2}\right)\left(-\frac{7}{2} + \frac{\sqrt{65}}{2}\right)^n$$

The two terms are not over $\mathbb{Q}$*, however they are conjugated and hence the sum* $u(n)$ *is defined over* $\mathbb{Q}$.

$$u(n) = [1, 1, -3, 25, -187, 1409, -10611, 79913, -601835, \ldots] \in \mathbb{Q}^{\mathbb{N}}$$

CHAPTER 3

MAIN ALGORITHM

In this thesis we introduce new algorithms that appear in Chapters 4, 5, and 6. The following algorithm calls these algorithms along with a short description of each call. As mentioned in Chapter 2 we focus on order 2. The remainder of this chapter contains material that is used in all three algorithms.

Algorithm Main:

Input: $L \in \mathbb{C}[n][\tau]$, a linear difference operator of order 2.

Let $L = a_2(n)\tau^2 + a_1(n)\tau + a_0(n)$.

Output: Solution of L, if we can find one from either existing software or one of the three algorithms introduced in this thesis.

1. Check currently available algorithms in CAS (Computer Algebra System) for a solution in the form of a factorization (i.e. a first order right hand factor was found) or some other general term solution. If existing algorithms find a solution then return that solution.

   Note: if no solutions were found then L is irreducible as existing software finds first order factors, if they exist ([2], [5], and [6]).

   If a complicated solution, other than a first order factor, is found by the CAS it may still be worthwhile to call steps 2, 3, and 4 to see if a simpler solution can be found.

2. Call Algorithm Find ${}_2F_1$ from Chapter 4.

   If we can find a gt-transformation $L \rightarrow L_2$, for L_2 a difference operator satisfied by a hypergeometric series $u(n) = {}_2F_1\left(\begin{matrix} a+n & b \\ & c \end{matrix} \middle| z \right)$ then return L_2 and the transformation.

3. Call Algorithm Find Liouvillian from Chapter 5.

If we can find a gt-transformation $L \to L_2$, for $L_2 = \tau^2 + \hat{b}(n)$ for some $\hat{b}(n) \in \mathbb{C}(n)$ then return L_2 and the transformation.

4. Call Algorithm Database Solver from Chapter 6.

   If we can find a gt-transformation $L \to L_2$, for L_2 a difference operator satisfied by a sequence from the Sloane database (OEIS) then return L_2 and the transformation.

In each of the introduced algorithms we need to be able to check if potential matches are related to the input by a gt-transformation. The remainder of this chapter will show that such a transformation (it will not be unique) will be found, if it exists.

Notation. *Given a polynomial $r(n)$ and a rational function $f(n) = p(n)/q(n)$, where we assume $p(n), q(n)$ to be relatively prime polynomials, we use notation $r(n) \mid f(n)$ or $r(n) \nmid f(n)$ to mean that $r(n) \mid p(n)q(n)$ or $r(n) \nmid p(n)q(n)$ respectively.*

Definition 3.1. *The* companion matrix *of a monic difference operator*

$$L = \tau^k + a_{k-1}\tau^{k-1} + \cdots + a_0,\ a_i \in \mathbb{C}(n)$$

which is satisfied by $u(n)$ will refer to the matrix:

$$M = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & \dots & -a_{k-2} & -a_{k-1} \end{pmatrix} \tag{3.1}$$

The equation $Lu = 0$ is equivalent to the system $\tau(Y) = MY$ where

$$Y = \begin{pmatrix} u(n) \\ \vdots \\ u(n+k-1) \end{pmatrix} \tag{3.2}$$

Definition 3.2. *Let $L = a_k\tau^k + a_{k-1}\tau^{k-1} + \cdots + a_0,\ a_i \in \mathbb{C}(n)$. The* determinant *of L,* $\det(L) := (-1)^k a_0/a_k$, *i.e. the determinant of its companion matrix.*

Definition 3.3. *The gauge transformation $G = c_{k-1}\tau^{k-1} + \cdots + c_0,\ c_i \in \mathbb{C}(n)$ operating on $u(n) \in V(L)$ is represented by the system:*

$$Z = \begin{pmatrix} Gu(n) \\ \tau(Gu(n)) \\ \vdots \\ \tau^{k-1}(Gu(n)) \end{pmatrix} \tag{3.3}$$

Which we can rewrite, using $Lu = 0$, as

$$Z = \hat{G}Y \tag{3.4}$$

with $\hat{G}$ a $k-$dimensional square matrix.

Lemma 3.1. *A gauge transformation $G\colon L_1 \to L_2$ maps:*

$$\det(L_1) \mapsto \det(L_1)\frac{\tau(\det(\hat{G}))}{\det(\hat{G})}$$

Proof. Let the companion matrix of $L_1u(n) = 0$ be M (so $\tau(Y) = MY$). The gauge transformation $Z = \hat{G}Y$ sends $M \mapsto \tau(\hat{G})M\hat{G}^{-1}$ by substituting

$$Y = \hat{G}^{-1}Z \text{ and } \tau(Y) = \tau(\hat{G}^{-1}Z) = \tau(\hat{G}^{-1})\tau(Z) = \tau(\hat{G})^{-1}\tau(Z)$$

in $\tau(Y) = MY$ to obtain

$$\tau(\hat{G})^{-1}\tau(Z) = M\hat{G}^{-1}Z \to \tau(Z) = \tau(\hat{G})M\hat{G}^{-1}Z$$

and the result follows. □

Lemma 3.2. *A gauge transformation G takes a second order irreducible recurrence equation to a second order irreducible recurrence equation.*

The proof of the Lemma follows from Definition 2.14.

Lemma 3.3. *If $L_1, L_2 \in \mathbb{C}(n)[\tau]$ are gauge equivalent of order k then there exist gauge equivalent $R, S \in \mathbb{C}(n)[\tau]$ of order $< k$ such that $L_1R = SL_2$.*

For a proof of the lemma in the differential case, which is adaptable to the difference case, see [9].

Lemma 3.4. *A t-transformation takes a second order irreducible recurrence equation to a second order irreducible recurrence equation.*

Proof. Let $f = a_2(n)u(n+2) + a_1(n)u(n+1) + a_0(n)u(n) = 0,\ a_i(n) \in \mathbb{C}(n)$. Let a t-transformation be defined by $v(n) = h(n)u(n),\ h(n)$ a hypergeometric sequence, then $\hat{a}_2(n)v(n+2) + \hat{a}_1(n)v(n+1) + \hat{a}_0(n)v(n) = 0$ where $\hat{a}_i(n) = a_i(n)/h(n+i) \in \mathbb{C}(n)$ so it just remains to show that $v(n)$ does not satisfy a first order recurrence equation, i.e. that $v(n)$ is not a hypergeometric solution. Suppose it is, i.e. suppose there exists $r_1(n) \in \mathbb{C}(n)$ such that

$$r_1(n) = \frac{v(n+1)}{v(n)} = \frac{h(n+1)u(n+1)}{h(n)u(n)}.$$

Then $r_1(n) = r_2(n)\frac{u(n+1)}{u(n)},\ r_2(n) \in \mathbb{C}(n) \ \Rightarrow \hat{r}(n) = u(n+1)/u(n),\ \hat{r}(n) \in \mathbb{C}(n)$ contradicting the irreducibility of f. □

Lemma 3.5. *If there exists a gauge transformation $L_1 \to L_2$ with L_1 irreducible then there exists a gauge transformation $L_2 \to L_1$*

The Lemma follows from the isomorphism between D/DL_1 and D/DL_2.

Lemma 3.6. *Let $\tau(Y) = MY,\ Y = \begin{pmatrix} u(n) \\ \vdots \\ u(n+k-1) \end{pmatrix}$.*
Under the gauge transformation given by $Z = \hat{G}Y$:

$$M \mapsto \tau(\hat{G})M\hat{G}^{-1} \tag{3.5}$$

Under the term product $v(n)u(n)$ with $v(n+1) - \alpha v(n) = 0,\ \alpha \in \mathbb{C}(n)$:

$$M \mapsto \tau(H)M(\det(H)H^{-1}),\ H = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix} \tag{3.6}$$

Lemma 3.7. *For any gauge transformation s_1 and t-transformation s_2 such that $f \xrightarrow{s_2 \circ s_1} g$ there exists a gauge transformation t_1 and a t-transformation t_2 such that $g \xrightarrow{t_1 \circ t_2} f$*

Proof. Let $S_2, T_2 = \begin{pmatrix} 1 & 0 \\ 0 & \tau(\alpha) \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\tau(\alpha)} \end{pmatrix}$ respectively, $\alpha \neq 0$ (i.e. $T_2 = S_2^{-1}$) and let M, N be the matrix representations for f, g respectively. From Equation (3.5):

$$\begin{aligned} N &= \tau(S_2)\tau(S_1)MS_1^{-1}(\alpha S_2^{-1}) \\ \tau(T_2)N(\frac{1}{\alpha}T_2^{-1}) &= \tau(T_2)\tau(S_2)\tau(S_1)MS_1^{-1}(\alpha S_2^{-1})(\frac{1}{\alpha}T_2^{-1}) \\ &= \tau(S_1)MS_1^{-1} \end{aligned}$$

By Lemma 3.5, there exists T_1 such that

$$M = \tau(T_1)\tau(T_2)N(\frac{1}{\alpha}T_2^{-1})T_1^{-1}$$

□

Lemma 3.8. *For $h(n)$ a hypergeometric term (i.e. $h(n+1)/h(n) \in \mathbb{C}(n)$), $h(n+k)/h(n) \in \mathbb{C}(n)$ for any $k \in \mathbb{Z}$.*

Proof. For $h(n) \in \mathbb{C}(n)$ the result is trivial so assume $h(n) \notin \mathbb{C}(n)$. For $k = 0$ we have $1 \in \mathbb{C}(n)$. For $k > 0$ we have

$$\frac{h(n+k)}{h(n)} = \frac{h(n+k)}{h(n+k-1)}\frac{h(n+k-1)}{h(n+k-2)}\cdots\frac{h(n+1)}{h(n)} = r_k r_{k-1}\cdots r_1,\ r_i \in \mathbb{C}(n)$$

For $k < 0$ we have

$$\frac{h(n)}{h(n+k)} = \frac{h(n)}{h(n-1)}\frac{h(n-2)}{h(n-1)}\cdots\frac{h(n+k-1)}{h(n+k)} = s_1 s_2\cdots s_k,\ s_i \in \mathbb{C}(n)$$

and so its reciprocal $h(n+k)/h(n) \in \mathbb{C}(n)$ □

Theorem 3.1. *If there exists a gauge transformation* $L_1 \to L_2 \otimes (\tau + a_1(n)),\ a_1(n) \in \mathbb{C}(n)$ *then for any* $a_2(n)$ *shift equivalent to* $a_1(n)$ *there exists a gauge transformation* $L_1 \to L_2 \otimes (\tau + a_2(n))$

Proof. Let $g(n)$ be a solution of L_2 and let $g_1(n) = h_1(n)g(n),\ g_2(n) = h_2(n)g(n)$ with $h_1(n), h_2(n)$ solutions of $\tau - a_1(n),\ \tau - a_2(n)$, respectively, then $g(n) = g_2(n)/h_2(n)$. For some non-negative integer k, $a_1(n+k) = a_2(n)$ or $a_1(n) = a_2(n+k)$ Without loss of generality, assume the former. $h_1(n+k)$ satisfies $\tau - a_1(n+k)$ $(= \tau - a_2(n))$.

Therefore $h_2(n) = \alpha h_1(n+k),\ \alpha \in \mathbb{C}$.

Given gauge transformation $g_1(n) \mapsto f(n)$:

$$\begin{aligned}
f(n) &= c_0(n)g_1(n) + c_1(n)g_1(n+1)\\
&= c_0(n)h_1(n)\frac{g_2(n)}{h_2(n)} + c_1(n)h_1(n+1)\frac{g_2(n+1)}{h_2(n+1)}\\
&= c_0(n)\frac{h_1(n)}{h_2(n)}g_2(n) + c_1(n)\frac{h_1(n+1)}{h_2(n+1)}g_2(n+1)\\
&= c_0(n)\frac{h_1(n)}{\alpha h_1(n+k)}g_2(n) + c_1(n)\frac{h_1(n+1)}{\alpha h_1(n+k+1)}g_2(n+1)\\
&= r_0(n)g_2(n) + r_1(n)g_2(n+1),\ r_0(n), r_1(n) \in \mathbb{C}(n)
\end{aligned}$$

□

Theorem 3.2. *Let the subscripts* $1, 2$ *or* $2, 1$ *denote g-transformations and t-transformations, respectively. Given* s_1, s_2 *there exist* t_1, t_2 *such that the following holds:*

Any gt-transformation $f \xrightarrow{s_2 \circ s_1} g$ *can be written as* $f \xrightarrow{t_1 \circ t_2} g$

Proof.

$$\begin{aligned}
v(n) &= c_0(n)(\Gamma(h(n))u(n)) + c_1(n)(\Gamma(h(n+1))u(n+1))\\
&= \Gamma(h(n))(c_0(n)u(n) + h(n)c_1(n)u(n+1))\\
&= \Gamma(h(n))(r_0(n)u(n) + r_1(n)u(n+1)),\ r_0(n), r_1(n), h(n) \in \mathbb{C}(n)
\end{aligned}$$

□

Corollary 3.1. *Let notation be the same as in Theorem 3.2 and let* $s_2 = (\tau - s(n))\otimes$ *and* $t_2 = (\tau - t(n))\otimes$*, then it follows that* $s(n) \overset{\text{SE}}{\equiv} \pm t(n)$.

Proof. Let $d = \det(f)$ then $f \xrightarrow{s_2 \circ s_1} g$ gives $\det(g) \overset{\text{SE}}{\equiv} s(n)s(n+1)d \overset{\text{SE}}{\equiv} (s(n))^2 d$.

Also $f \xrightarrow{t_1 \circ t_2} g$ gives $\det(g) \overset{\text{SE}}{\equiv} t(n)t(n+1)d \overset{\text{SE}}{\equiv} (t(n))^2 d$.

Therefore $t(n) \overset{\text{SE}}{\equiv} \pm s(n)$ □

Theorem 3.3. *Let* $s_1, \ldots, s_m$ *be some combination of gauge transformations and t-transformations. A transformation* $f \xrightarrow{s_1 \circ \ldots \circ s_m} g$ *can be written* $f \xrightarrow{t_2 \circ t_1} g$ *for some gauge transformation* t_1 *and some t-transformation* t_2.

Proof. By Theorem 3.2 we may assume that $s_1, \ldots, s_j$ are gauge transformations and $s_{j+1}, \ldots, s_m$ are t-transformations and then all that is left to show is:

1. The composition of any number of gauge transformations can be written as a single gauge transformation:

   Consider gauge transformations of the form $Z = GY$ where $\tau(Y) = MY$. The composition of j gauge transformations is given by the product of matrices:

   $Y_2 = G_1 Y_1,\ Y_3 = G_2 Y_2,\ \ldots,\ Y_j = G_{j-1} Y_{j-1},\ G_i \in \mathrm{GL}_2(\mathbb{C}(n))$

   therefore $Y_j = GY_1,\ G \in \mathrm{GL}_2(\mathbb{C}(n))$

   Note: Given $Z = GY,\ f \mapsto h$: if $G \in MAT_{\alpha\times\alpha}(\mathbb{C}(n))$ and $G \notin \mathrm{GL}_\alpha(\mathbb{C}(n))$ then f is reducible (see proof of Lemma 3.5).

2. The composition of any number of successive t-transformations can be written as a single t-transformation:

$$u_m(n) = h_{m-1}(n)u_{m-1}(n)$$
$$\vdots$$
$$u_3(n) = h_2(n)u_2(n)$$
$$u_2(n) = h_1(n)u_1(n)$$

   $\Rightarrow u_m(n) = h(n)u_1(n)$ with $h(n) = h_{m-1}(n)\cdots h_1(n)$ and so $\frac{h(n+1)}{h(n)} \in \mathbb{C}(n)$

□

Theorem 3.4. *If there exists a gt-transformation $L_1 \to L_2$ then there exists a gauge transformation $L_1 \otimes (\tau - r) \to L_2$, where $r = \pm\sqrt{\mathrm{SNF}(\det(L_2)/\det(L_1))}$*

Proof. L_2 and $L_1 \otimes (\tau - r)$ have the same determinant up to shift equivalence. The result follows from the proof of Corollary 3.1. □

We summarize how we find a gt-transformation between two operators L_1 and L_2 :

Theorem 3.4 reduces the problem of finding a gt-transformation to the problem of finding a gauge transformation (we try both the ± cases: $r = \pm\sqrt{\mathrm{SNF}(\det(L_2)/\det(L_1))}$ from Theorem 3.4). We next employ the following two algorithms:

Algorithm Find gt-Transformation:
Input: $L_1, L_2 \in \mathbb{C}[n][\tau]$ linear difference operators of order 2.
Output: Operator of the form $H(n)(c_1(n)\tau + c_0(n))$ mapping $V(L_1)$ to $V(L_2)$.

1. Calculate $\hat{r} = \mathrm{SNF}(\det(L_2)/\det(L_1))$.
2. If $\hat{r}$ is a square in $\mathbb{C}(x)$ then let $r = \sqrt{\hat{r}}$ else return 'FAIL' and stop.
3. Calculate $L_{neg} = L_1 \otimes (\tau - r)$ and $L_{pos} = L_1 \otimes (\tau + r)$.
4. Call Algorithm Find Gauge Transformation with arguments L_2, L_{neg}.
   (a) If result is not 'FAIL' then return $H(n)\cdot$ result and exit, where $H(n)$ is a solution of $(\tau - r)$.
5. Call Algorithm Find Gauge Transformation with arguments L_2, L_{pos}.
   (a) If result is not 'FAIL' then return $H(n)\cdot$ result and exit, where $H(n)$ is a solution of $(\tau + r)$.
   (b) If result is 'FAIL' then return 'FAIL.'

It is known that finding a gauge transformation can be reduced to finding rational solutions of a system and the latter can be done using e.g. the EG-elimination algorithm from [10]. For completeness, we outline the steps taken.

Note: The procedure from [10] that we use limits us to working over $\mathbb{Q}(n)$ (see Theorem 2.2).

Algorithm Find Gauge Transformation:
Input: $L_a, L_b \in \mathbb{C}[n][\tau]$ linear difference operators of order 2. Let $u(n) \in V(L_a),\ v(n) \in V(L_b)$.
Output: Operator of the form $c_1(n)\tau + c_0(n)$ mapping $V(L_b)$ to $V(L_a)$.

1. Write $u(n) = c_0(n)v(n) + c_1(n)v(n+1)$ (where c_0, c_1 are to be determined) and make appropriate substitutions (from $L_b(v(n) = 0)$ so that $u(n), u(n+1), u(n+2) \in \mathbb{Q}(n)[v(n), v(n+1)]$.

2. Substitute $u(n) = c_0(n)v(n) + c_1(n)v(n+1)$ and its shifts (τ and τ^2) into L_a, set the coefficients of $v(n), v(n+1)$ equal to zero and solve for $c_0(n+2), c_1(n+2)$. Assign to $c_0(n+2), c_1(n+2)$ these solutions.

3. $c_0(n+2), c_1(n+2) \in \mathbb{Q}(n)[G]$, for $G = [c_0(n), c_1(n), c_0(n+1), c_1(n+1)]$. Let the coefficients of G in $c_0(n+2)$ be $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, respectively and $\beta_1, \beta_2, \beta_3, \beta_4$, respectively, for $c_1(n+2)$.

4. Call [10] to solve the system $\tau(Y) = MY$ for Y where:

$$Y = \begin{pmatrix} c_0(n) \\ c_0(n+1) \\ c_1(n) \\ c_1(n+1) \end{pmatrix}, \quad M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ 0 & 0 & 0 & 1 \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 \end{pmatrix} \tag{3.7}$$

5. If the trivial solution is returned for Y then return 'FAIL' otherwise return $Y[3]\tau + Y[1]$.

CHAPTER 4

GENERALIZED HYPERGEOMETRIC SERIES

4.1 Preliminaries

Definition 4.1. *The* generalized hypergeometric series ${}_mF_n$ *is a power series in* z *with* m *upper parameters and* n *lower parameters:*

$$ {}_mF_n\left(\begin{matrix} a_1 & \dots & a_m \\ b_1 & \dots & b_n \end{matrix}\middle| z\right) = \sum_{k\geqslant 0} \frac{(a_1)_k \cdots (a_m)_k}{(b_1)_k \cdots (b_n)_k} \frac{z^k}{k!} \tag{4.1} $$

(Recall the Pochhammer symbol from Definition 2.11.)

Proposition 4.1. ${}_mF_n$ *is undefined when any lower parameter is a non-positive integer without the existence of a corresponding upper parameter that is a non-positive integer of lesser or equal absolute value.*

Proposition 4.2. *If* ${}_mF_n$ *is defined and* $m = n + 1$ *then Equation (4.1) converges for all* $|z| < 1$. *(This case and convergence for the cases* $m < n + 1,\ m > n + 1$ *are seen upon application of the ratio test.)*

Note: ${}_mF_n$ is considered a generalized hypergeometric series (i.e. if $m = 2, n = 1$ then omit the word "generalized"). When a hypergeometric series is convergent it will define a hypergeometric function.

Example 4.1. ${}_3F_2\left(\begin{matrix} -3 & 9/5 & -1 \\ -2 & -4 \end{matrix}\middle| \frac{1}{2}\right)$ *is defined, but* ${}_2F_1\left(\begin{matrix} 2/3 & -3 \\ -1 \end{matrix}\middle| \frac{1}{2}\right)$ *is not defined.*

${}_2F_1$ hypergeometric series are used to represent many elementary functions. As an example consider the following hypergeometric series:

$$ {}_2F_1\left(\begin{matrix} n & 1 \\ 1 \end{matrix}\middle| z\right) = 1 + nz + \frac{(n)_2}{2!}z^2 + \frac{(n)_3}{3!}z^3 + \frac{(n)_4}{4!}z^4 + O(z^5) $$

We replace n, z by $-n, -z$ respectively to make the right hand side look more familiar:

$$ {}_2F_1\left(\begin{matrix} -n & 1 \\ & 1 \end{matrix}\middle| -z\right) = 1 + nz + \frac{(n-1)_2}{2!}z^2 + \frac{(n-2)_3}{3!}z^3 + \frac{(n-3)_4}{4!}z^4 + O(z^5) $$

We see that the right hand side is the series expansion of $(1+z)^n$ and so:

$$ (1+z)^n = {}_2F_1\left(\begin{matrix} -n & 1 \\ & 1 \end{matrix}\middle| -z\right),\ z \in \mathbb{C} $$

In this case we notice that the ${}_2F_1$ function can trivially be represented as a ${}_1F_0$ function.

4.1.1 Contiguous Relations

Following are identities due to Gauss, Pfaff, and Euler respectively:

$$ a(z-1)\,{}_2F_1\left(\begin{matrix} a+1 & b \\ & c \end{matrix}\middle| z\right) + (a(2-z)+bz-c)\,{}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right) + (c-a)\,{}_2F_1\left(\begin{matrix} a-1 & b \\ & c \end{matrix}\middle| z\right) = 0 \quad (4.2) $$

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right) = (1-z)^{-a}\,{}_2F_1\left(\begin{matrix} a & c-b \\ & c \end{matrix}\middle| \frac{z}{z-1}\right),\ z \notin [1,\infty) \quad (4.3) $$

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right) = (1-z)^{c-a-b}\,{}_2F_1\left(\begin{matrix} c-a & c-b \\ & c \end{matrix}\middle| z\right) \quad (4.4) $$

Note: Equation (4.3) is equivalent to:

$$ {}_2F_1\left(\begin{matrix} a & b \\ & c \end{matrix}\middle| z\right) = (1-z)^{-b}\,{}_2F_1\left(\begin{matrix} c-a & b \\ & c \end{matrix}\middle| \frac{z}{z-1}\right),\ z \notin [1,\infty) \quad (4.5) $$

Definition 4.2. *Two ${}_2F_1$ hypergeometric series are* contiguous *when their corresponding upper and lower parameters differ by integers.*

Example 4.2.

$$ \frac{bz}{c}{}_2F_1\left(\begin{matrix} a+n+1 & b+1 \\ & c+1 \end{matrix}\middle| z\right) = {}_2F_1\left(\begin{matrix} a+n+1 & b \\ & c \end{matrix}\middle| z\right) - {}_2F_1\left(\begin{matrix} a+n & b \\ & c \end{matrix}\middle| z\right) \quad (4.6) $$

This identity is proved with straightforward calculations which we include as an example.

Proof.

$$ {}_2F_1\left(\begin{matrix} a+n+1 & b \\ & c \end{matrix}\middle| z\right) = 1 + \frac{(a+n+1)b}{c}z + \frac{(a+n+1)(a+n+2)b(b+1)}{c(c+1)}\frac{z^2}{2} + \cdots $$

$$ {}_2F_1\left(\begin{matrix} a+n & b \\ & c \end{matrix}\middle| z\right) = 1 + \frac{(a+n)b}{c}z + \frac{(a+n)(a+n+1)b(b+1)}{c(c+1)}\frac{z^2}{2} + \cdots $$

After subtracting we obtain, for the j'th non-zero term:

$$ (a+n+j-a-n)\frac{(a+n+1)\cdots(a+n+j)b\cdots(b+j)}{c\cdots(c+j)}\frac{z^j}{j!} = \frac{bz}{c}\frac{(a+n+1)_{j-1}(b+1)_{j-1}}{(c+1)_{j-1}}\frac{z^{j-1}}{(j-1)!} $$

and the result follows. ◻

Other relations we will use follow (after replacement of a with $a+n$) and may be obtained in the same manner, from sources such as http://functions.wolfram.com, or, using Equation (4.7) as an example, by entering the command '$Contiguous2f1([0,0,0],[0,1,0],[1,0,0]);$' into the Maple package for computing contiguous relations of Gauss hypergeometric 2F1 series written by R. Vidunas [8].

$$(a-b)\,{}_2F_1\left(\begin{matrix}a\ \ b\\ c\end{matrix}\middle|z\right)+b\,{}_2F_1\left(\begin{matrix}a\ \ b+1\\ c\end{matrix}\middle|z\right)-a\,{}_2F_1\left(\begin{matrix}a+1\ \ b\\ c\end{matrix}\middle|z\right)=0 \tag{4.7}$$

$$(b-c)\,{}_2F_1\left(\begin{matrix}a\ \ b-1\\ c\end{matrix}\middle|z\right)-a(z-1)\,{}_2F_1\left(\begin{matrix}a+1\ \ b\\ c\end{matrix}\middle|z\right)+(c-a-b)\,{}_2F_1\left(\begin{matrix}a\ \ b\\ c\end{matrix}\middle|z\right)=0 \tag{4.8}$$

$$c(a+(b-c)z)\,{}_2F_1\left(\begin{matrix}a\ \ b\\ c\end{matrix}\middle|z\right)+(a-c)(b-c)z\,{}_2F_1\left(\begin{matrix}a\ \ b\\ c+1\end{matrix}\middle|z\right)+ac(z-1)\,{}_2F_1\left(\begin{matrix}a+1\ \ b\\ c\end{matrix}\middle|z\right)=0 \tag{4.9}$$

$$(c-1)\,{}_2F_1\left(\begin{matrix}a\ \ b\\ c-1\end{matrix}\middle|z\right)-a\,{}_2F_1\left(\begin{matrix}a+1\ \ b\\ c\end{matrix}\middle|z\right)+(a-c+1)\,{}_2F_1\left(\begin{matrix}a\ \ b\\ c\end{matrix}\middle|z\right)=0 \tag{4.10}$$

Theorem 4.1. *For any integers k,l,m there exist unique functions $P_{klm}, Q_{klm} \in \mathbb{Q}(a,b,c,z)$ such that*

$$ {}_2F_1\left(\begin{matrix}a+k\ \ b+l\\ c+m\end{matrix}\middle|z\right)=P_{klm}\,{}_2F_1\left(\begin{matrix}a\ \ b\\ c\end{matrix}\middle|z\right)+Q_{klm}\,{}_2F_1\left(\begin{matrix}a+1\ \ b\\ c\end{matrix}\middle|z\right) \tag{4.11}$$

An algorithm to compute these P_{klm} and Q_{klm} as well as a proof of the theorem is given in [8].

4.1.2 Solutions of the base equation

In Equation (4.2), we substitute $a+n+1$ for a and find the operator of minimal order satisfied by the sequence $u(n) = {}_2F_1\left(\begin{matrix}a+n\ \ b\\ c\end{matrix}\middle|z\right)$.

$$L_{\text{base}} = (a+n+1)(z-1)\tau^2 + ((a+n+1)(2-z)+bz-c)\tau + (c-a-n-1) \tag{4.12}$$

L_{base} will be the base equation in the chapter; subsection 4.2 will give an algorithm to decide if an equation is solvable in terms of solutions of L_{base} for some $a,b,c,z \in \mathbb{C}$. Subsection 4.3 will then spell out the solution of L in terms of ${}_2F_1$ hypergeometric series.

Definition 4.3. *The* Casoratian determinant *of n operators is the determinant of the* Casoratian matrix*, C, which is the matrix that has entries $c_{i,j} = \tau^i(f_j)_{j=1..n}^{i=0..n-1}$.*

Remark 4.1. *Irreducibility of Equation (4.12) is not implied, e.g. $u(n) = {}_2F_1\left(\begin{matrix}a+n\ \ -1\\ c\end{matrix}\middle|z\right) = (c-za-zn)/c$ satisfies a first order recurrence operator (similarly, the result is a polynomial for any non-positive integer b). Since our aim is to solve irreducible equations, the cases $b \in [0,-1,-2,\dots]$ are therefore not of interest to this program.*

4.2 Algorithm Find ${}_2F_1$

Algorithm Find ${}_2F_1$:

Input: $L \in \mathbb{C}[n][\tau]$ a second order, irreducible, homogeneous recurrence relation. Let $L = a_2(n)\tau^2 + a_1(n)\tau + a_0(n)$

Output: gt-transformation from $L_{\text{base}} = (a+n+1)(z-1)\tau^2+((a+n+1)(2-z)+bz-c)\tau+(c-a-n-1)$ to L, if such a transformation exists.

1. If $2 \cdot \deg(a_1(n)) > \deg(a_2(n)) + \deg(a_0(n))$ or $\deg(a_2(n)) - \deg(a_0(n))$ is not even then return 'FAIL' and stop.

2. Write the shift normalized determinant of L as $kd\left(\hat{d}\right)^2$ with d a square free polynomial, $\hat{d} \in \mathbb{C}(n)$, $k \in \mathbb{C}$, and $d, \hat{d}$ monic.

   (a) If d is quadratic then let the roots of d be α_1, α_2 and let $c = \alpha_2 - \alpha_1$.

   (b) If $d = 1$ then let $c = 1$.

   (c) If $d \neq 1$ and d is not quadratic then return 'FAIL' and stop.

3. Let $\hat{L} = L \otimes \left(\tau + 1/\hat{d}\right) = \tau^2 + b_1(n)\tau + b_0(n),\ b_0(n), b_1(n) \in \mathbb{C}(n)$ and let $\hat{b}_1(n) = \text{SNF}(b_1(n))$.

4. If $c \neq 1$ then let $a = -\alpha_1$ and continue at Step 5.

   If $c = 1$ then let $dt =$ the denominator of $\hat{b}_1(n)$. For r a root of dt, let $a = -r$, and continue at Step 5.

5. Let $y_1, y_2 \in \mathbb{C}((1/n))$ equal the solutions of $y^2 + b_1(n)y + b_0(n) = 0$ and, since $b_0(n) \neq 0$, $\{y_1/y_2, y_2/y_1\} = \left\{\sum_{i=0}^{\infty} c_{i1} (1/n)^i, \sum_{i=0}^{\infty} c_{i2} (1/n)^i\right\}$.

   Compute $c_{01}, c_{11}, c_{02}, c_{12}$ and for each $j \in \{1, 2\}$ do the following:

   (a) Let $z = 1 - c_{0j}$ and let $\hat{b} = \frac{1}{2}\left(c + c_{1j}/c_{0j}\right)$.

   (b) For $b \in \{\hat{b}, \hat{b} + 1/2\}$ search for a gauge transformation from L_{base} to $\hat{L}$ and, if successful, calculate the gt-transformation from L_{base} to L (i.e. call Algorithm Find Gauge Transformation).

6. If $c \neq 1$ and Step 5 is unsuccessful then return 'FAIL' and stop. (See Remark 4.3.)

   If $c = 1$ and Step 5 is unsuccessful then repeat Step 4 with $r =$ the next untested root of dt. If Step 5 is unsuccessful after all roots of dt have been tested then return 'FAIL' and stop.

7. Call Algorithm Find Independent Solutions in section 4.3 and, if successful, return the gt-transformation obtained in Step 5 along with the basis of solutions obtained from the call. If the call is unsuccessful then return the gt-transformation with the solution ${}_2F_1\left(\begin{smallmatrix} a+n & b \\ c \end{smallmatrix}\middle| z\right)$.

4.2.1 Algorithm Step 1

Idea: Rule out candidates that cannot be transformations of ${}_2F_1$ functions. A series of the form ${}_2F_1\left(\begin{smallmatrix} a+n & b \\ c \end{smallmatrix}\middle| z\right)$ satisfies Equation (4.12).

Lemma 4.1. *Given a sequence $u(n)$, for which $u(n+1)/u(n)$ is not a rational function, there exists at most one (up to equivalence, see Remark 2.1) second order homogeneous recurrence relation satisfied by $u(n)$.*

Proof. Suppose there exist two such recurrence equations, divide each by their respective leading coefficients (coeff$(u(n+2))$) :

$$f_1 : u(n+2) = p_1(n)u(n+1) + q_1(n)u(n)$$

$$f_2 : u(n+2) = p_2(n)u(n+1) + q_2(n)u(n)$$

If $p_1(n) = p_2(n)$ then it must also be that $q_1(n) = q_2(n)$ (from $f_1 - f_2$) and so $f_1 = f_2$.

If $p_1(n) \neq p_2(n)$ then $(p_1(n) - p_2(n))u(n+1) = (q_2(n) - q_1(n))u(n)$ leading to $u(n+1)/u(n) = (q_2(n) - q_1(n))/(p_1(n) - p_2(n)) \in \mathbb{C}(n)$, a contradiction of the assumption. □

Definition 4.4. *Let v_∞ be the valuation on $\mathbb{C}(n)$ such that:*

1. $v_\infty(0) = \infty$

2. $v_\infty(f) = \deg(\text{denominator}(f), n) - \deg(\text{numerator}(f), n)$, *for* $f \neq 0$

Remark. *When working modulo squares $g(n) \equiv 1/g(n)$, for $g(n) \in \mathbb{C}(n) - \{0\}$. So, modulo squares, roots are equivalent to poles and only roots/poles of odd multiplicity count. Consider the determinant of our operator:*

$$L = \tau^2 + \frac{a_1(n)}{a_2(n)}\tau + \frac{a_0(n)}{a_2(n)}$$

$$\det(L) = \frac{a_0(n)}{a_2(n)} \equiv \frac{a_2(n)}{a_0(n)} \equiv a_0(n)a_2(n) \ \textit{modulo squares}$$

The use of Newton Polygons, starting from Equation (4.12), explains the first condition of Step 1, the second condition can be explained either by Newton Polygons or by the determinant being invariant under gauge transformations modulo shift equivalence and being invariant under t-transformations modulo shift equivalence and modulo squares.

Definition 4.5. *Let $L \in \mathbb{C}[n][\tau],\ L = \sum a_i \tau^j$. The* Newton polygon *of L, is defined as*

$$N(L) = \textit{Convex Hull}(\{(i, j) \in \mathbb{Z}^2 \mid j \geqslant v_\infty(a_i)\})$$

Properties:

1. $N(L_1L_2) = N(L_1) + N(L_2) = \{(x, y) + (\tilde{x}, \tilde{y}) \mid (x, y) \in N(L_1), (\tilde{x}, \tilde{y}) \in N(L_2)\}$

2. $N(L_1) + N(R) = N(L_2) + N(R) \Rightarrow N(L_1) = N(L_2)$

Lemma 4.2. *Let $L_1, L_2 \in \mathbb{C}(n)[\tau], L_1 = \sum_{k=0}^{d} a_k(n)\tau^k, L_2 = \sum_{k=0}^{d} b_k(n)\tau^k,\ a_o, b_0, a_d, b_d \neq 0$. If L_1, L_2 are gauge equivalent then $N(L_1) = N(L_2) + \{(0, v_\infty(a_d) - v_\infty(b_d))\}$,*
i.e. $N(L_1) \equiv N(L_2)$ modulo an up or down shift of the convex hull.

Proof. We prove the Lemma by induction.

The Lemma is true for the case $d = 0$, we assume the case $d = k$ is true and show that it is then true for the case $d = k + 1$.

Let L_1, L_2 be gauge equivalent of order $k + 1$. Consider $L_1R = S L_2$ for R, S of order $< k$. Such R, S exist and are gauge equivalent by the gauge equivalence of L_1, L_2 (see Corollary 3.3). Then:

$$N(L_1) + N(R) = N(S) + N(L_2) \Rightarrow N(L_1) + N(R) = N(R) + \{(0, c)\} + N(L_2),\ c \in \mathbb{Z}$$

(by the induction assumption) and so $N(L_1) = N(L_2) + \{(0, c)\}$ by Property 2 above. □

Idea: Under gt-transformations the determinant, modulo shift equivalence and modulo squares, is invariant, i.e. our transformations cannot introduce new factors into the determinant. This invariant helps determine the parameters a and c. Under certain conditions the roots of a quadratic equation obtained from L_{base} will give information that is invariant under our transformations and will help determine the parameters b and z.

4.2.2 Algorithm Steps 2-4

Proposition 4.3. *Assume a gt-transformation from the input relation to one satisfied by a* ${}_2F_1\left(\begin{smallmatrix} a+n & b \\ & c \end{smallmatrix} \middle| z\right)$ *exists. This* ${}_2F_1$ *satisfies Equation (4.12) and:*

- *Under gauge transformations the determinant is invariant modulo shift equivalence.*
- *Under term products the determinant modulo squares is invariant modulo shift equivalence.*

The Proposition is a restatement of Lemmas 3.1 and 3.6.

Lemma 4.3. L_{base} *is gauge equivalent to each of the following:*

1. *substitute* $a+k$ *for* a *in* L_{base} $(k \in \mathbb{Z})$
2. *substitute* $b+k$ *for* b *in* L_{base} $(k \in \mathbb{Z})$
3. *substitute* $c+k$ *for* c *in* L_{base} $(k \in \mathbb{Z})$

Proof. We prove the case $k = 1$ and the general case $k \in \mathbb{Z}$ follows immediately from induction and from the invertibility of gauge transformations (for when $k < 0$). Let $f(n) = {}_2F_1\left(\begin{smallmatrix} a+n & b \\ & c \end{smallmatrix} \middle| z\right)$.

Gauge equivalence of L_{base} with 1 follows from the definition of gauge equivalence, i.e.

$${}_2F_1\left(\begin{smallmatrix} a+1+n & b \\ & c \end{smallmatrix} \middle| z\right) = f(n+1)$$

To show gauge equivalence of L_{base} with 2 we use the identities Equation (4.7) and Equation (4.8). After substituting $a+n$ for a we obtain

$${}_2F_1\left(\begin{smallmatrix} a+n & b+1 \\ & c \end{smallmatrix} \middle| z\right) = \frac{b-a-n}{b} f(n) + \frac{a+n}{b} f(n+1) \tag{4.13}$$

To show gauge equivalence of L_{base} with 3 we use the identities Equation (4.9) and Equation (4.10). After substituting $a+n$ for a we obtain

$${}_2F_1\left(\begin{smallmatrix} a+n & b \\ & c+1 \end{smallmatrix} \middle| z\right) = -\frac{c(a+n+(b-c)z)}{(a+n-c)(b-c)z} f(n) + \frac{(a+n)c(1-z)}{(a+n-c)(b-c)z} f(n+1) \tag{4.14}$$

□

Let there exist a gt-transformation from $L \in \mathbb{C}[n][\tau]$ to an operator of the form:

$$\tilde{L} = (\tilde{a}+n+1)(\tilde{z}-1)\tau^2 + ((\tilde{a}+n+1)(2-\tilde{z}) + \tilde{b}\tilde{z} - \tilde{c})\tau + (\tilde{c} - \tilde{a} - n - 1)$$

We show that we can recover parameters $a, b, c, z \in \mathbb{C}$ such that there is a gt-transformation to L from:

$$(a+n+1)(z-1)\tau^2 + ((a+n+1)(2-z) + bz - c)\tau + (c-a-n-1) \tag{4.15}$$

(i.e. another, not necessarily distinct, operator of the form L_{base}) by examining the two possible cases:

1. $\tilde{c} \notin \mathbb{Z}$, i.e. modulo shift equivalence the determinant of L has two root(s)/pole(s) with odd multiplicity

2. $\tilde{c} \in \mathbb{Z}$, i.e. modulo shift equivalence the determinant of L has no roots or poles with odd multiplicity

Up to shift equivalence and modulo squares, $\det(L) = \det(\tilde{L}) = \dfrac{\tilde{c} - \tilde{a} - n}{\tilde{a} + n}$ and so the above list is a complete list of possible cases (see Proposition 4.3). Note that the root and pole of $\det(\tilde{L})$ are $n = \tilde{c} - \tilde{a}$ and $n = -\tilde{a}$ modulo shift equivalence.

Case 1: ($\tilde{c} \notin \mathbb{Z}$)

Let $d = \text{SNF}(\det(L))$ modulo squares. If d is quadratic (modulo squares we write f/g as $\hat{f}\hat{g}$ with $\hat{f}, \hat{g}$ monic) then let $d = (n+a)(n+a-c)$. Taking the difference of the roots leads to a choice, for a, c, of:

Case 1a: $\begin{Bmatrix} a = \tilde{a} + n_a \\ c = \tilde{c} + n_c \end{Bmatrix}$ with $n_a, n_c \in \mathbb{Z}$.

or

Case 1b: $\begin{Bmatrix} a = \tilde{a} - \tilde{c} + n_a \\ c = -\tilde{c} + n_c \end{Bmatrix}$ with $n_a, n_c \in \mathbb{Z}$.

These cases are considered after Theorem 4.3.

Case 2: ($\tilde{c} \in \mathbb{Z}$) Let $c = 1$. After the term product in Step 3 let dt = the denominator of the coefficient of τ in L. We show that, modulo shift equivalence, $n + a$ is a factor of dt. Now proceed as in Case 1a for each a corresponding to a factor of dt.

4.2.3 Algorithm Step 5

Idea: We look at power series about $n = \infty$ of the ratio of solutions of L_{base}. We use the property that the first two terms in the expansion are invariant modulo shifts of n to find invariants from which we recover the parameters b, z. The power series can get more complicated with t-transformations so we avoid this problem with the t-transformation in Step 3.

Definition 4.6. *Given two polynomials A, B with $v_\infty(A) = v_\infty(B)$ we say that $A \overset{k}{\sim} B$ when $v_\infty(A - B) \geqslant v_\infty(A) + k$.*

Definition 4.7. *Let $L, M \in \mathbb{C}(n)[\tau]$ each have order k and let a_i^L, a_i^M be their respective coefficients of τ^i. We say that $L \overset{2}{\sim} M$ when $v_\infty(a_i^L - a_i^M) \geqslant v_\infty(a_i^L) + 2,\ i = 0..k$.*

Notation. *$\diamond$ refers to commutative product, e.g. $(\tau + r(n)) \diamond (\tau + s(n)) = \tau^2 + (r(n) + s(n))\tau + r(n)s(n)$.*

Lemma 4.4. *Let $y_1, y_2 \in \mathbb{C}[[n^{-1}]]$ with $v_\infty(y_1) = v_\infty(y_2) = v_\infty(y_1 - y_2) = 0$ and let*

$$L_1 = (\tau - y_1) \diamond (\tau - y_2)$$

$$L_2 = (\tau - y_1)(\tau - y_2)$$

$$L_3 = \mathrm{LCLM}(\tau - y_1, \tau - y_2)$$

then $L_1 \overset{2}{\sim} L_2 \overset{2}{\sim} L_3$.

The Lemma can be proved with straightforward computations.

Remark 4.2. *Solving*

$$(a + n + 1)(z - 1)y^2 + ((a + n + 1)(2 - z) + bz - c)y + (c - a - n - 1) = 0 \tag{4.16}$$

produces $y_1, y_2 \in \mathbb{C}[[n^{-1}]]$ such that $L_{base} = (\tau - y_1) \diamond (\tau - y_2)$. The discriminant of Equation (4.16), up to one equivalence, is z^2 and the case of $z = 0$ is already excluded (the hypergeometric series is reduced to the constant 1) thus $y_1 \overset{1}{\nsim} y_2$ and, up to interchanging y_1, y_2, we have

$$\begin{aligned} \frac{y_1}{y_2} &= 1 - z + \frac{(1 - z)(c - 2b)}{n} + o\left(n^{-2}\right) \\ \frac{y_2}{y_1} &= \frac{1}{1 - z} + \frac{2b - c}{(1 - z)n} + o\left(n^{-2}\right) \end{aligned} \tag{4.17}$$

Theorem 4.2. *Let $L = (\tau - y_1) \diamond (\tau - y_2)$. There exist $\tilde{y}_1, \tilde{y}_2 \in \mathbb{C}[[n^{-1}]]$ with $y_1 \overset{2}{\sim} \tilde{y}_1$ and $y_2 \overset{2}{\sim} \tilde{y}_2$ such that $L = \mathrm{LCLM}(\tau - \tilde{y}_1, \tau - \tilde{y}_2)$.*

Proof. Let $\tilde{L} \overset{2}{\sim} (\tau - y_1)(\tau - y_2)$. Hensel lifting gives $y_1^k, y_2^k \in \mathbb{C}[[n^{-1}]]$ such that $\tilde{L} \overset{k}{\sim} (\tau - y_1^k)(\tau - y_2^k)$ for $k = 3, 4, 5, \ldots$

Let $\tilde{y}_2 = \lim_{k\to\infty} y_2^k$.

We also have $\tilde{L} \overset{2}{\sim} (\tau - y_1)\diamond(\tau - y_2) = (\tau - y_2)\diamond(\tau - y_1) \overset{2}{\sim} (\tau - y_2)(\tau - y_1)$ by Lemma 4.4. Hensel lifting now gives $\hat{y}_1^k, \hat{y}_2^k \in \mathbb{C}[[n^{-1}]]$ such that $\tilde{L} \overset{k}{\sim} (\tau - \hat{y}_2^k)(\tau - \hat{y}_1^k)$ for $k = 3, 4, 5, \ldots$

Let $\tilde{y}_1 = \lim_{k\to\infty} \hat{y}_1^k$. □

Lemma 4.5. *Let $L, \tilde{y}_1, \tilde{y}_2$ be as in Theorem 4.2. If there exists a gauge transformation from L to L_{base} then applying the gauge transformation to $\tau - \tilde{y}_1$ and $\tau - \tilde{y}_2$ gives $\tilde{z}_1, \tilde{z}_2 \in \mathbb{C}[[n^{-1}]]$ with $\tilde{z}_1 \overset{2}{\nsim} \tilde{z}_2$ such that*

$$L = \mathrm{LCLM}(\tau - \tilde{z}_1, \tau - \tilde{z}_2)$$

and thus

$$L \overset{2}{\sim} (\tau - \tilde{z}_1)\diamond(\tau - \tilde{z}_2)$$

The proof of the Lemma follows from Lemma 4.4 and from $y_1 \overset{1}{\nsim} y_2$ (see Remark 4.2).

Theorem 4.3. *Let*

- $y_1, y_2, z_1, z_2 \in \mathbb{C}[[n^{-1}]]$.
- $L_{base} = \mathrm{LCLM}(\tau - y_1, \tau - y_2)$
- *there exists a gt-transformation $L_{base} \to L = \mathrm{LCLM}(\tau - z_1, \tau - z_2)$*
- $\frac{y_1}{y_2} = c_0 + \frac{c_1}{n} + o\left(n^{-2}\right)$

Then, up to multiplicative inverse,

$$\frac{y_1}{y_2} \mapsto \frac{z_1}{z_2} = c_0 + \frac{c_1 + kc_0}{n} + o\left(n^{-2}\right),\ k \in \mathbb{Z}$$

under gt-transformations.

Proof. A term product $L \otimes (\tau - a)$ is equal to $\mathrm{LCLM}(\tau - z_1 a, \tau - z_2 a)$ and so z_1/z_2 is invariant under term products. The proof is now reduced to considering z_1/z_2 under gauge transformations where $\tau - y_1 \overset{GE}{\sim} \tau - z_1$ and $\tau - y_2 \overset{GE}{\sim} \tau - z_2$.

Consider the gauge transformation $r_0 + r_1\tau$ with $r_0, r_1 \in \mathbb{C}(n)$ sending $u(n) \to v(n)$ where $u(n+1) - g(n)u(n) = 0$ and $v(n+1) - h(n)v(n) = 0$. Note that $g(n), h(n) \in \mathbb{C}[[n^{-1}]]$. Let $\hat{g}(n) \in \mathbb{C}(n)$ with $\hat{g}(n) \overset{2}{\sim} g(n)$ and substitute:

$$v(n) = r_0(n)u(n) + r_1(n)u(n+1) \text{ and } v(n+1) = r_0(n+1)u(n+1) + r_1(n+1)u(n+2)$$

into

$$v(n+1) - h(n)v(n) = 0$$

and solve for $h(n)$:

$$\begin{aligned} h(n) &= \frac{g(n)(r_0(n+1) + r_1(n+1)g(n+1))}{r_0(n) + r_1(n)g(n)} \\ &\overset{2}{\sim} g(n)\frac{r_0(n+1) + r_1(n+1)\hat{g}(n+1)}{r_0(n) + r_1(n)\hat{g}(n)} \overset{2}{\sim} g(n)\frac{\tau(s(n))}{s(n)},\ s(n) \in \mathbb{C}(n) \end{aligned} \tag{4.18}$$

and so, up to interchanging y_1, y_2:

$$y_1 \overset{2}{\sim} z_1\frac{s_1}{\tau(s_1)} = z_1\left(1 + \frac{k_1}{n} + o\left(n^{-2}\right)\right),\ k_1 \in \mathbb{Z}$$

$$y_2 \overset{2}{\sim} z_2\frac{s_2}{\tau(s_2)} = z_2\left(1 + \frac{k_2}{n} + o\left(n^{-2}\right)\right),\ k_2 \in \mathbb{Z}$$

thus,

$$\frac{y_1}{y_2} \overset{2}{\sim} \frac{z_1}{z_2}\left(1 + \frac{k}{n} + o\left(n^{-2}\right)\right),\ k = k_1 - k_2$$

and the conclusion follows. □

Note, from the right hand sides of Equations (4.17), that the integer shifts $b \mapsto b+k,\ c \mapsto c+k$ or $b \mapsto b-k,\ c \mapsto c-k$ send $c_1 \to c_1 + kc_0$, i.e. in practice, other than realizing that b, c can be shifted by an integer in this step, we may disregard k when searching for b, c

Recall that $\tilde{L} = (\tilde{a}+n+1)(\tilde{z}-1)\tau^2 + ((\tilde{a}+n+1)(2-\tilde{z}) + \tilde{b}\tilde{z} - \tilde{c})\tau + (\tilde{c} - \tilde{a} - n - 1)$.

Let y_1, y_2 be the solutions of $(\tilde{a}+n+1)(\tilde{z}-1)y^2 + ((\tilde{a}+n+1)(2-\tilde{z}) + \tilde{b}\tilde{z} - \tilde{c})y + (\tilde{c} - \tilde{a} - n - 1) = 0$.

Let $\left\{s, \frac{1}{s}\right\} = \left\{\frac{y_1}{y_2}, \frac{y_2}{y_1}\right\}$ and calculate their series expansions in $\mathbb{C}[[n^{-1}]]$:

$$\left\{s, \frac{1}{s}\right\} = \left\{1 - \tilde{z} + \frac{(\tilde{c} - 2\tilde{b})(1-\tilde{z})}{n} + o\left(n^{-2}\right),\ \frac{1}{1-\tilde{z}} + \frac{2\tilde{b} - \tilde{c}}{(1-\tilde{z})n} + o\left(n^{-2}\right)\right\}$$

For $[s_1, s_2] = [s, \frac{1}{s}]$ and $[s_1, s_2] = [\frac{1}{s}, s]$ we calculate:

- $1 - \text{coeff}(s_1, n^0) = \begin{cases} 1 - (1-\tilde{z}) = \tilde{z} \\ 1 - \dfrac{1}{1-\tilde{z}} = \dfrac{\tilde{z}}{\tilde{z}-1} \end{cases}$

- $\frac{1}{2}\left(c + \frac{\text{coeff}(s_2, n^{-1})}{\text{coeff}(s_2, n^0)}\right) = \begin{cases} \dfrac{1}{2}\left(c + (\tilde{c} - 2\tilde{b})\right) \\ \dfrac{1}{2}\left(c + (2\tilde{b} - \tilde{c})\right) \end{cases}$

Giving us two subcases for each of Case 1a and Case 1b:

$$\textbf{Case 1a:} \begin{cases} a = \tilde{a} + j_1 \\ b = \tilde{c} - \tilde{b} + j_2/2 \\ c = \tilde{c} + j_3 \\ z = \tilde{z}/(\tilde{z} - 1) \end{cases} \quad \text{or} \quad \begin{cases} a = \tilde{a} + j_4 \\ b = \tilde{b} + j_5/2 \\ c = \tilde{c} + j_6 \\ z = \tilde{z} \end{cases} \quad \text{with } j_i \in \mathbb{Z}.$$

$$\textbf{Case 1b:} \begin{cases} a = \tilde{a} - \tilde{c} + k_1 \\ b = -\tilde{b} + k_2/2 \\ c = -\tilde{c} + k_3 \\ z = \tilde{z}/(\tilde{z} - 1) \end{cases} \quad \text{or} \quad \begin{cases} a = \tilde{a} - \tilde{c} + k_4 \\ b = \tilde{b} - \tilde{c} + k_5/2 \\ c = -\tilde{c} + k_6 \\ z = \tilde{z} \end{cases} \quad \text{with } k_i \in \mathbb{Z}.$$

(Note that an integer shift of parameters a, b, or c is just a g-transformation and so we need only be concerned with the half-integer shifts, i.e. we need to check both $\tilde{b}$ and $\tilde{b} + 1/2$.)

Remark 4.3. *Theorem 4.4 shows that the map* $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a+1-c \\ b+1-c \\ 2-c \end{pmatrix}$ *sends* $\tilde{L}$ *from Equation (4.12) to a term product of* $\tilde{L}$ *and so* $\begin{pmatrix} a \\ b \\ c \end{pmatrix} \mapsto \begin{pmatrix} a-c \\ b-c \\ -c \end{pmatrix}$ *sends* $\tilde{L}$ *to a gt-transformation of* $\tilde{L}$*. Thus working in either case 1a or case 1b is sufficient.*

4.3 Algorithm Find Independent Solutions

For the theorem below we exclude certain cases when considering $f_1(n) = {}_2F_1\left(\begin{matrix} a+n & b \\ & c \end{matrix} \middle| z\right)$ and $f_2(n) = \frac{\Gamma(a+n+1-c)}{\Gamma(a+n)} \, {}_2F_1\left(\begin{matrix} a+n+1-c & b+1-c \\ & 2-c \end{matrix} \middle| z\right)$.

- If $b \in \{0, -1, -2, \dots\}$ then L_{base} is reducible (see Remark 4.1).
- If $c = 1$ then $f_1(n) = f_2(n)$. The remaining cases of $c \in \mathbb{Z}$ are considered:
  - If $c \in \{0, -1, -2, \dots\}$ and $b \notin \{0, -1, \dots, c\}$ then $f_1(n)$ is undefined.

    (If $c \in \{0, -1, -2, \dots\}$ and $b \in \{0, -1, \dots, c\}$ then L_{base} is reducible.)

– If $c \in \{2, 3, 4, \dots\}$ and $b \notin \{1, 2, \dots, c-1\}$ then $f_2(n)$ is undefined.

(If $c \in \{2, 3, 4, \dots\}$ and $b \in \{1, 2, \dots, c-1\}$ then L_{base} is reducible.)

Note: We did not consider cancellation with $a+n$ because, for large enough n, there will still be division by zero.

In Summary, if $c \notin \mathbb{Z}$, $z \neq 1$, and $b \notin [0, -1, -2, \dots]$ then the theorem will provide two independent solutions.

Theorem 4.4. *Let* $f_1(n) = {}_2F_1\left(\begin{matrix} a+n & b \\ & c \end{matrix} \middle| z\right)$, $f_2(n) = \frac{\Gamma(a+n+1-c)}{\Gamma(a+n)}\, {}_2F_1\left(\begin{matrix} a+n+1-c & b+1-c \\ & 2-c \end{matrix} \middle| z\right)$*. If* $f_1(n), f_2(n)$ *are defined*[1] *and* $c, z \neq 1$ *then* $f_1(n)$ *and* $f_2(n)$ *are linearly independent solutions of Equation* (4.12).

Proof. f_1, f_2 are seen to be solutions after application of Equation (4.2).

To prove linear independence, consider the Casoratian determinant, C, of $\{f_1(n), f_2(n)\}$:

$$C = \begin{vmatrix} f_1(n) & f_2(n) \\ f_1(n+1) & f_2(n+1) \end{vmatrix} = \frac{{}_2F_1\left(\begin{matrix} a+n & b \\ & c \end{matrix} \middle| z\right)\Gamma(a+n+2-c)\, {}_2F_1\left(\begin{matrix} a+n+2-c & b+1-c \\ & 2-c \end{matrix} \middle| z\right)}{\Gamma(a+n+1)} -$$

$$\frac{\Gamma(a+n+1-c)\, {}_2F_1\left(\begin{matrix} a+n+1-c & b+1-c \\ & 2-c \end{matrix} \middle| z\right)\, {}_2F_1\left(\begin{matrix} a+n+1 & b \\ & c \end{matrix} \middle| z\right)}{\Gamma(a+n)}$$

Let $r = \frac{dC}{dz}/C$. Using contiguous relations we can simplify r to $\frac{a+b+n+1-c}{1-z}$. Let $\hat{C} = (1-z)^{c-a-b-n-1}$ then $\hat{C}'/\hat{C} = r = C'/C$ so $C/\hat{C}$ must be independent of z, hence

$$C = C_{|z=0}\hat{C} = \frac{(1-c)\Gamma(a+n+1-c)}{\Gamma(a+n+1)}(1-z)^{c-a-b-n-1}$$

From this form of C it is clear, for $c, z \neq 1$, that $C \neq 0$. (The case $z = 1$ defines a first order recurrence equation and so $f_1 = f_2$ defines the solution space of L.) □

For a discussion on the use of the Casoratian determinant to check for linear independence in the difference case see Appendix A in [7].

Definition 4.8. *The* Greatest Common Right Divisor *of two operators,* GCRD(L_1, L_2)*, is the operator whose solution space is equal to* $V(L_1) \cap V(L_2)$*. It can be found with the right Euclidean algorithm.*

Definition 4.9. *The* Least Common Left Multiple *of two operators,* LCLM(L_1, L_2)*, is the operator whose solution space is equal to* $V(L_1) + V(L_2)$*. It can be found by solving* $\mathbb{C}-$*linear equations.*

[1] We need $a+n+1-c$ and $a+n \notin \{0, -1, -2, \dots\}$ so that the Γ functions are defined; see Proposition 4.1 for conditions on ${}_mF_n$

Algorithm Find Independent Solutions:

Input: Parameters a, b, c, z from $L_{\text{base}} = (a+n+1)(z-1)\tau^2 + ((a+n+1)(2-z)+bz-c)\tau + (c-a-n-1) \in \mathbb{C}[n][\tau]$

Output: Linearly independent solutions (given explicitly in Theorem 4.4).

1. Compute $f_2(n)$ from Theorem 4.4, if it exists (see explanation preceding theorem and the conditions of the theorem).

2. Use identities (on both $f_1(n)$ and $f_2(n)$ simultaneously, if previous step was successful) to try to obtain 'nicer' function(s) $\hat{f}_i(n)$ (see Remark 4.4).

3. Return $\hat{f}_1(n)$ and, if the first step was successful, $\hat{f}_2(n)$.

Remark 4.4. *We make minor efforts to return a 'nicer' output by trying Pfaff transformations (if allowed) as well as trying shifts of* ± 1 *and* ± 2 *for each parameter repeatedly until we no longer decrease character count (e.g. using Maple's length command). Recall that any integer shift(s) of the parameter(s) will be a gauge transformation as evidenced by the identities we use:*

param	shift	identity: ${}_2F_1\left(\left.{a+n,\ b \atop c}\right\| z\right) =$
a	-1	$\frac{-2a+az-2n+nz+2-z-bz+c}{(z-1)(a+n-1)}\,{}_2F_1\left(\left.{a+n-1,\ b \atop c}\right\| z\right) + \frac{-c+a+n-1}{(z-1)(a+n-1)}\,{}_2F_1\left(\left.{a+n-2,\ b \atop c}\right\| z\right)$
a	$+1$	$-\frac{-2a+az-2n+nz-2+z-bz+c}{1-c+a+n}\,{}_2F_1\left(\left.{a+n+1,\ b \atop c}\right\| z\right) + \frac{(z-1)(a+n+1)}{1-c+a+n}\,{}_2F_1\left(\left.{a+n+2,\ b \atop c}\right\| z\right)$
b	-1	$\frac{b-1-a-n}{b-1}\,{}_2F_1\left(\left.{a+n,\ b-1 \atop c}\right\| z\right) + \frac{a+n}{b-1}\,{}_2F_1\left(\left.{a+n+1,\ b-1 \atop c}\right\| z\right)$
b	$+1$	$\frac{(a+n)(z-1)}{b+1-c}\,{}_2F_1\left(\left.{a+n+1,\ b+1 \atop c}\right\| z\right) - \frac{c-a-n-b-1}{b+1-c}\,{}_2F_1\left(\left.{a+n,\ b+1 \atop c}\right\| z\right)$
c	-1	$-\frac{(c-1)(a+n+(b-c+1)z)}{(a+n-c+1)(b-c+1)z}\,{}_2F_1\left(\left.{a+n,\ b \atop c-1}\right\| z\right) + \frac{(c-1)(a+n)(1-z)}{(a+n-c+1)(b-c+1)z}\,{}_2F_1\left(\left.{a+n+1,\ b \atop c-1}\right\| z\right)$
c	$+1$	$\frac{a+n}{c}\,{}_2F_1\left(\left.{a+n+1,\ b \atop c+1}\right\| z\right) - \frac{a+n-c}{c}\,{}_2F_1\left(\left.{a+n,\ b \atop c+1}\right\| z\right)$
z	$-$	$(1-z)^{-a-n}\,{}_2F_1\left(\left.{a+n,\ c-b \atop c}\right\| \frac{z}{z-1}\right),\ z \notin [1, \infty)$
z	$-$	$(1-z)^{-b}\,{}_2F_1\left(\left.{c-a-n,\ b \atop c}\right\| \frac{z}{z-1}\right),\ z \notin [1, \infty)$

The above identities were obtained from Equations (4.7)-(4.10), (4.3), and L_{base}. The identities for shifts of the parameters by ± 2 are then easily obtained using L_{base}.

Example 4.3. *We input* L_{base} *with parameters* $a, b, c, z = 3/2, -11/7, -6/5, 1/4$. *Before using these identities the output, for k constant, is:*

$$u(n) = k(4/3)^n \left(-21(17+10n)\,{}_2F_1\left(\left.{1/2+n,\ 13/35 \atop -1/5}\right| -1/3\right) + 5(42n-1)\,{}_2F_1\left(\left.{3/2+n,\ 13/35 \atop -1/5}\right| -1/3\right)\right) \quad (4.19)$$

and after trying to find a nicer output:

$$u(n) = k\ {}_2F_1\left(\begin{matrix} 3/2+n & & -11/7 \\ & -6/5 & \end{matrix}\middle| 1/4\right)$$

CHAPTER 5

LIOUVILLIAN

Definition 5.1 (Definition 6 in [3])**.** *Let* $L_1, L_2 \in \mathbb{C}(n)[\tau]$. *The* symmetric product *of* L_1 *and* L_2 *is defined as the monic operator* $L \in \mathbb{C}(n)[\tau]$ *of smallest order such that* $L(u_1u_2) = 0$ *for all* $u_1, u_2 \in S$ *with* $L_1u_1 = 0$ *and* $L_2u_2 = 0$.

Notation. *The* symmetric square *of L, denoted* $L^{\circledS 2}$*, will refer to the symmetric product of L and L (i.e. with itself).*

Liouvillian solutions are defined in [7] Section 3.2. For irreducible operators they are characterized by the following theorem (see Propositions 31-32 in [4]).

Theorem 5.1. *An irreducible k'th order operator L has* Liouvillian *solutions if and only if its companion matrix is gauge equivalent to one that can be written as*

$$M = \begin{pmatrix} 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 1 \\ -a & 0 & \dots & 0 & 0 \end{pmatrix}, \ a \in \mathbb{C}(n)$$

In other words, L is gauge equivalent to $\tau^k + a$.

Remark. u^2, uv *will both refer to the term by term commutative product.*

Lemma 5.1. *Let* $L = a_2\tau^2 + a_1\tau + a_0,\ a_i \in \mathbb{C}[n],\ a_0, a_2 \neq 0.$

1. *If* $a_1 \neq 0$ *then*

$$L^{\circledS 2} = c_3\tau^3 + c_2\tau^2 + c_1\tau + c_0,$$

$$\begin{aligned} \textit{where:}\ \ c_3 &= a_1(n)a_2(n+1)^2a_2(n) \\ c_2 &= a_1(n+1)a_2(n)(-a_1(n+1)a_1(n) + a_0(n+1)a_2(n)) \\ c_1 &= -a_0(n+1)a_1(n)(-a_1(n+1)a_1(n) + a_0(n+1)a_2(n)) \\ c_0 &= -a_1(n+1)a_0(n+1)a_0(n)^2 \end{aligned} \tag{5.1}$$

2. *If* $a_1 = 0$ *then*

$$L^{\circledS 2} = a_2^2\tau^2 - a_0^2 \tag{5.2}$$

$$L^{\circledS 2} \textit{ has order: } \begin{cases} 2, & \textit{if } a_1 = 0 \\ 3, & \textit{if } a_1 \neq 0 \end{cases}$$

Proof. 1. (For the case $a_1 = 0$ we solve $Lu(n) = 0$ for $u(n+2)$ and square both sides.)

For the case $a_1 \neq 0$ we solve $Lu(n+1) = 0$ for $u(n+3)$, expand its square, and for any term not containing $u(n+2)^2$ make the substitution $u(n+2) = -\frac{a_1}{a_2}u(n+1) - \frac{a_0}{a_2}u(n)$ if possible. $L^{\circledS 2}(u(n)^2)$ (from Equation (5.1)) is obtained after the further substitution

$$u(n)u(n+1) = \frac{u(n+2)^2 - a(n)^2u(n+1)^2 - b(n)^2u(n)^2}{2a(n)b(n)} \tag{5.3}$$

(Equation (5.3) comes from the expansion of $u(n+2)^2 = (-a(n)u(n+1) - b(n)u(n))^2$.)

If $Lu_1 = 0, Lu_2 = 0$ then it is easy to check that u_1u_2 satisfies Equation (5.1) (or Equation (5.2) if $a_1 = 0$). To prove that Equation (5.1) respectively (5.2) is indeed $L^{\circledS 2}$ we need to prove that their orders are minimal, i.e. we need to prove the second part of the Lemma.

2. Let $a_1 = 0$. We can find $u_1, u_2 \in \mathbb{C}^{\mathbb{N}}$ such that for some $q \in \mathbb{N}$ and q larger than any integer root of a_0a_n :
$$\begin{pmatrix} u_1(q) & = 0 & u_1(q+1) & = 1 \\ u_2(q) & = 1 & u_2(q+1) & = 0 \end{pmatrix}$$
With these u_1, u_2 we see that u_1^2, u_2^2 are linearly independent and so $V(L^{\circledS 2}) \subseteq V(a_2^2\tau^2 - a_0^2)$ is an equality.

Let $a_1 \neq 0$. For $u_1, u_2 \in V(L)$:

$$L = r\begin{vmatrix} u_1(n) & u_1(n+1) & u_1(n+2) \\ u_2(n) & u_2(n+1) & u_2(n+2) \\ 1 & \tau & \tau^2 \end{vmatrix},\ r = \frac{a_2}{M_{3,3}}$$

($M_{3,3}$ is the $(3,3)$ minor.)

After expansion and factoring it can be seen that the following determinant equals $a_0a_1a_2 \neq 0$ up to some nonzero constant:

$$D = \begin{vmatrix} u_1(n)^2 & u_1(n+1)^2 & u_1(n+2)^2 \\ u_2(n)^2 & u_2(n+1)^2 & u_2(n+2)^2 \\ u_1(n)u_2(n) & u_1(n+1)u_2(n+1) & u_1(n+2)u_2(n+2) \end{vmatrix} \tag{5.4}$$

Therefore u_1^2, u_2^2, u_1u_2 are $\mathbb{C}-$linearly independent which implies that the order of $L^{\circledS 2}$ is at least 3 and hence equal to 3.

□

Remark. *The proof of the below Lemma illustrates computations in Step 4 of the following Algorithm. Using difference modules the proof of a more general lemma follows from the isomorphism between the difference modules defined by $L, \hat{L}$.*

Lemma 5.2. *Let $a \neq 0$. Given a gauge transformation from $L = \tau^2 + a(n)\tau + b(n)$ to $\hat{L} = \tau^2 + r(n)$ one can compute a difference operator mapping $V(L^{\circledS 2})$ onto $V(\hat{L}^{\circledS 2})$.*

Proof. Let $u(n) \in V(L)$ and $v(n) = g_0(n)u(n) + g_1(n)u(n+1) \in V(\hat{L})$
then $v(n)^2 = g_0(n)^2u(n)^2 + 2g_0(n)g_1(n)u(n)u(n+1) + g_1(n)^2u(n+1)^2$.
The substitution (obtained by squaring $u(n+2) = -a(n)u(n+1) - b(n)u(n)$):

$$u(n)u(n+1) = \frac{u(n+2)^2 - a(n)^2u(n+1)^2 - b(n)^2u(n)^2}{2a(n)b(n)}$$

yields:

$$v(n)^2 = \frac{g_0(n)(-g_1(n)b(n) + g_0(n)a(n))}{a(n)}u(n)^2 - \frac{g_1(n)(-g_1(n)b(n) + g_0(n)a(n))}{b(n)}u(n+1)^2 + \frac{g_0(n)g_1(n)}{a(n)b(n)}u(n+2)^2 \tag{5.5}$$

□

Algorithm Find Liouvillian:

Input: $L \in \mathbb{C}[n][\tau]$ a second order, irreducible, homogeneous recurrence operator.

Let $L = a_2(n)\tau^2 + a_1(n)\tau + a_0(n)$.

Output: A two-term recurrence operator, $\hat{L}$, with a gauge transformation from $\hat{L}$ to L, if it exists.

1. If $a_1 = 0$ then return $\hat{L} = L$ and stop.

2. Let $u(n)$ be an indeterminate function. Impose the relation $Lu(n) = 0$, i.e.
$$u(n+2) = -\frac{1}{a_2(n)}(a_0(n)u(n) + a_1(n)u(n+1)) \tag{5.6}$$

3. Let R be a non-zero rational solution of $L^{\circledS 2} \otimes (\tau + 1/\det(L))$, if such a solution exists, else return 'FAIL' and stop.

4. Let g be an indeterminate and let $v(n) = gu(n) + u(n+1)$. Compute $d_0, d_1, d_2 \in \mathbb{C}(n)[g]$ such that
$$v(n)^2 = d_0 u(n)^2 + d_1 u(n+1)^2 + d_2 u(n+2)^2 \tag{5.7}$$
(To compute d_0, d_1, d_2 first substitute Equation (5.6) into Equation (5.7).)

5. Let S denote a non-zero solution of $\tau + \det(L)$, so $\tau(S) = -\det(L)S$. Then $u(n)^2$ and RS satisfy $L^{\circledS 2}$, see steps 2 and 3. Substitute the following
$$\begin{aligned} u(n)^2 &= RS \\ u(n+1)^2 &= -R(n+1)\det(L)S \\ u(n+2)^2 &= R(n+2)\tau(\det(L))\det(L)S \end{aligned} \tag{5.8}$$
into Equation (5.7) to get $v(n)^2 = SA$ for some $A \in \mathbb{C}(n)[g]$.

6. Solve $A = 0$ for g and choose one solution. A is a quadratic equation so this solution is in $\mathbb{C}(n)$ or in a quadratic extension of $\mathbb{C}(n)$. If $g \notin \mathbb{C}(n)$ then return an error message and stop.

7. Return $\hat{L}$ as well as the transformation $V(\hat{L}) \to V(L)$:
$$\hat{L} = \tau^2 + b(n)\frac{\tau(\delta)}{\delta} \tag{5.9}$$
$$u(n) = \frac{1}{\delta}((g(n+1) - a(n))v(n) - v(n+1)) \tag{5.10}$$
where $a(n) = a_1/a_2$, $b(n) = a_0/a_2$, $\delta = g(n)g(n+1) - g(n)a(n) + b(n)$, and $v(n)$ denotes a solution of $\hat{L}$.

Idea: Assume that $L = \tau^2 + a_1\tau + a_0,\ a_1 \neq 0$ is Liouvillian and irreducible, then there exists a gauge transformation $L \to \hat{L}$ for some $\hat{L}$ of the form $\tau^2 + r$ (see Theorem 5.1). $\dim(V(L^{\circledS 2})) = 3$ and $\dim(V(\hat{L}^{\circledS 2})) = 2$ so the transformation $L^{\circledS 2} \to \hat{L}^{\circledS 2}$ has a 1−dimensional kernel corresponding to a right hand factor of $L^{\circledS 2}$ (i.e. $L^{\circledS 2}$ has a hypergeometric solution, namely RS in Step 5). This gives us the gauge transformation from $V(L)$ to $V(\hat{L})$. In order that Step 3 only needs to search for a solution $\in \mathbb{C}(n)$ (which is easier than computing a more general hypergeometric solution) we work with the term product of $L^{\circledS 2}$ and $\tau - 1/\det(L)$ and use Theorem 5.2. The gauge transformation given in Step 7 was found by computing the inverse of the gauge transformation $v(n) = gu(n) + u(n+1)$ introduced in Step 4 (where the value of g is computed in Steps 5 and 6).

Example 5.1. *Consider Sloane sequence* A081123 = floor$(n/2)!$. *Algorithm Find Liouvillian will return the relationship* A081123$(n) = \frac{2}{n-1}v(n) + \frac{2}{n}v(n+1)$ *for a* $v(n) \in V(\tau^2 - \frac{(n+1)^2}{2(n-1)})$.

5.1 Algorithm Step 3

Let L be from **Input** and let $\hat{L}$ be a two-term operator.

Remark 5.1. *A t-transformation of a two-term operator is again a two-term operator. It follows that if there exists a gt-transformation from L to $\hat{L}$ then we may disregard the t-transformation, i.e. we need only search for a gauge transformation.*

Lemma 5.3. *Let* $A_1, A_2, G_s \in \mathbb{C}(n)[\tau],\ \mathrm{ord}(A_1) = 3,\ \mathrm{ord}(A_2) = 2$, *and assume that* $G_s(V(A_1)) = V(A_2)$, *i.e.* $V(A_1) \overset{G_s}{\twoheadrightarrow} V(A_2)$. *Then* A_1 *has a first order right hand factor, as well a a second order left hand factor that is gauge equivalent to* A_2.

Proof. $V(\mathrm{GCRD}(A_1, G_s)) = V(A_1) \cap V(G_s) = \ker(G_s\colon V(A_1) \twoheadrightarrow V(A_2))$ which has dimension $3 - 2 = 1$ and so A_1 has a first order right hand factor, $L_1 = \mathrm{GCRD}(A_1, G_s)$. Write $A_1 = L_2L_1,\ G_s = \tilde{G}L_1$ then $\tilde{G}\colon V(L_2) \to V(A_2)$ shows that L_2 is gauge equivalent to A_2. □

(We now substitute $L^{\circledS 2}, \hat{L}^{\circledS 2}$ for A_1, A_2, respectively, from the preceding Lemma.)

Theorem 5.2. *Let* $\hat{L} = r_2\tau^2 + r_0$ *and let* $L = a_2\tau^2 + a_1\tau + a_0,\ a_i \neq 0$. *Suppose there exists a gauge transformation* $G\colon V(L) \to V(\hat{L})$ *then by Lemma 5.2 there is* $G_s\colon V(L^{\circledS 2}) \twoheadrightarrow V(\hat{L}^{\circledS 2})$. *Let* $L_1 = \mathrm{GCRD}(G_s, L^{\circledS 2})$ *(which has order* 1 *by Lemma 5.3) and write* $L^{\circledS 2} = L_2L_1$. *Then*

$$\det(L_1) \overset{\mathrm{SE}}{\equiv} -\det(L)$$

Proof. 1. $\det(L) \overset{\text{SE}}{\equiv} \det(\hat{L})$, see Lemma 3.1.

2. $\det(L^{\circledS 2}) \overset{\text{SE}}{\equiv} \det(L)^3$, see the formula for $L^{\circledS 2}$ in Lemma 5.1.

3. $\det(\hat{L}^{\circledS 2}) = -\det(\hat{L})^2 \overset{\text{SE}}{\equiv} -\det(L)^2$, see Lemma 5.1 with $a_1 = 0$ and Item 1, respectively.

4. $\det(L^{\circledS 2}) = \det(L_2 L_1) \overset{\text{SE}}{\equiv} \det(L_2)\det(L_1)$.

5. L_2 is gauge equivalent to $\hat{L}^{\circledS 2}$, see Lemma 5.3.

6. $\det(L_2) \overset{\text{SE}}{\equiv} \det(\hat{L}^{\circledS 2}) = -\det(\hat{L})^2 \overset{\text{SE}}{\equiv} -\det(L)^2$, see Items 5,3,1.

7. $\det(L_1) \overset{\text{SE}}{\equiv} -\det(L)$, see Items 4,2,6.

□

Corollary 5.1. *Let $L, \hat{L}, G$ be as in Theorem 5.2 so that $L^{\circledS 2} = L_2 L_1$ then there exists a rational solution of $L^{\circledS 2} \otimes (\tau + 1/\det(L))$.*

Step 3 computes this rational solution.

5.2 Algorithm Step 4

Lemma 5.4. *If $T_1 \colon g_1\tau + g_0$ with $g_1 \neq 0$ defines a gauge transformation from L to a two-term operator then $T_2 \colon \tau + g_0/g_1$ is also a gauge transformation from L to a two-term operator.*

The two transformations differ by the t-transformation $u \mapsto g_1 u$ and so the Lemma's claim follows from Remark 5.1. (The case $g_1 = 0$ defines a t-transformation.)

5.3 Algorithm Step 5

Equation (5.7) defines the map $L^{\circledS 2} \to \hat{L}^{\circledS 2}$ and RS is in the kernel. (RS is the hypergeometric solution of $L^{\circledS 2}$ that we computed using Corollary 5.1.)

5.4 Algorithm Step 6

It can be proven that if there exists a gauge transformation from $L \in \mathbb{C}(n)[\tau]$ to an operator of the form $\tau^2 + \tilde{a}$ where $\tilde{a}$ is algebraic over $\mathbb{C}(n)$ then there also exists a gauge transformation $G \colon L \to \tilde{L} = \tau^2 + a$ with $G, \tilde{L} \in \mathbb{C}(n)[\tau]$. Note that if $L \in C(n)[\tau]$, $C \subset \mathbb{C}$ then an algebraic extension of C may occur, see the following example.

Example 5.2. *$L_1 = \tau^2 - \tau - (n^2 + 1)$ and $L_2 = \tau^2 - (n + i)(n + 1 - i)$ are gauge equivalent with $L_1 \in \mathbb{Q}(n)[\tau]$ and $L_2 \in \mathbb{Q}(n)[\tau, \sqrt{-1}]$. Both $\frac{1}{n-i}\tau + 1$, which sends L_2 to L_1, and its inverse are $\in \mathbb{C}(n)[\tau]$.*

CHAPTER 6

DATABASE SOLVER

Idea: The On-Line Encyclopedia of Integer Sequences (OEIS) maintained by Neil J. A. Sloane (see [1]) contains more than 140, 000 sequences and gives many pieces of information about these sequences such as references, formulas, related sequences, etcetera. When output is given in terms of a sequence from Sloane's database, e.g. for the sequence named A000085, the user will find information about this sequence at World-Wide Web URL www.research.att.com/~njas/sequences/A000085. We only use those that satisfy second order irreducible operators.

Algorithm Database: Try to solve in terms of a known sequence.
Input: $L \in \mathbb{C}[n][\tau]$, a linear difference operator of order 2 and, optionally, a solution in $\mathbb{C}^{\mathbb{N}}$ of L.
Let $L = a_2(n)\tau^2 + a_1(n)\tau + a_0(n)$
Output: Solutions of the form $H(n)(c_0(n)u(n) + c_1(n)u(n+1))$ such that
$c_0(n), c_1(n), H(n+1)/H(n) \in \mathbb{C}(n),\ u(n)$ a sequence from the Sloane database (OEIS) if such a solution exists.

1. Compute the p-curvature constants (see Definition 6.1) for a pre-determined number of primes p and check for matches among representatives of all groupings of Sloane sequences.

2. For each match, $\hat{L}$, compute the ratio (of the input to the potential match) of shift normalized determinants until the following subitems are true (pass the first such match along or return Fail if there are no matches):

   (a) The ratio is a square in $\mathbb{C}(n) - \{0\}$.

   (b) There exists a gauge transformation from the input to $\hat{L} \otimes (\tau \pm r)$, where r is a square root of the ratio (i.e. call Algorithm Find gt-Transformation).

3. **If** input includes a solution (i.e. a difference equation with initial conditions defining a sequence)

   then check that the match is a Sloane sequence or a linear combination of two Sloane sequences (a second order difference equation has two linearly independent solutions/sequences). If no such match exists then return 'FAIL'

   else return result with arbitrary constants.

6.1 Algorithm Step 1

Definition 6.1. *The* p-curvature[1] *of a matrix* M *is given by* $\tau^{p-1}(M)\cdots\tau(M)M$. *The* p-curvature *of a difference operator will refer to the* p-curvature *of its companion matrix* M *(see Definition 3.1).*

Definition 6.2. *For a kth order difference operator the* p-curvature constant *will refer to* a_{k-1}^{k}/a_0 *where the* a_i *come from the characteristic polynomial,* $\lambda^k + a_{k-1}\lambda^{k-1} + \cdots + a_0\lambda^0$, *of the p-curvature modulo p for each p of a predetermined set (in our case* $\{2, 3, 5\}$*).*

Theorem 6.1. *The characteristic polynomial (charpoly) of the p-curvature of a difference operator modulo p is invariant under gauge transformations.*

Proof. Let a gauge transformation of $\tau(Y) = MY$ with p-curvature $\tilde{M}$ be given by $\hat{Y} = AY$.

Then $\tau(\hat{Y}) = \tau(A)\tau(Y) = \tau(A)MY = \tau(A)MA^{-1}\hat{Y}$

and so the p-curvature after a gauge transformation is

$$\tau^p(A)\tau^{p-1}(M)\tau^{p-1}(A^{-1})\tau^{p-1}(A)\tau^{p-2}(M)\cdots\tau(A^{-1})\tau(A)MA^{-1} =$$

$$\tau^p(A)\tau^{p-1}(M)\tau^{p-2}(M)\cdots\tau(M)MA^{-1} \equiv$$

$$A\tilde{M}A^{-1} \mod p$$

and therefore

$$\text{charpoly}\left((A\tilde{M})A^{-1}\right) = \text{charpoly}\left(A^{-1}A\tilde{M}\right) = \text{charpoly}\left(\tilde{M}\right)$$

□

Theorem 6.2. *The p-curvature constant is invariant under t-transformations.*

[1] See [6] for more on the use of p-curvature

Proof. Under a t-transformation $Y \mapsto \hat{Y}$ with $\tau(Y) = MY$ and p-curvature (of Y) $\tilde{M}$

$M \mapsto \alpha\tau(H)MH^{-1}, \; H = \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$ see Equation (3.6).

So the p-curvature after a t-transformation is

$$\tau^{p-1}(\alpha)\tau^{p}(H)\tau^{p-1}(M)\tau^{p-1}(H^{-1})\tau^{p-2}(\alpha)\tau^{p-1}(H)\tau^{p-2}(M)\cdots\tau(H^{-1})\alpha\tau(H)MH^{-1} =$$

$$(\alpha)_p\tau^{p}(H)\tau^{p-1}(M)\tau^{p-2}(M)\cdots\tau(M)MH^{-1} \equiv$$

$$(\alpha)_p H\tilde{M}H^{-1} \mod p$$

and therefore

$$\text{charpoly}\left((\alpha)_p H\tilde{M}H^{-1}\right) = \text{charpoly}\left((\alpha)_p HH^{-1}\tilde{M}\right) = \text{charpoly}\left((\alpha)_p\tilde{M}\right)$$

$$\lambda^{k} + (\alpha)_p a_{k-1}\lambda^{k-1} + \cdots + (\alpha)_p^{k} a_0$$

□

The field of constants of $\mathbb{F}_p(n)$ under the automorphism $\mathbf{T}$ given by $\mathbf{T}f = \tau(f)\mathbf{T}$ is $\mathbb{F}_p(n^p - n)$ ([6]).

Remark 6.1. *Since a shift is a gauge transformation, Theorem 6.1 tells us that the a_i are constants (i.e. invariant under the action of τ).*

Coefficients of the characteristic polynomial of the p-curvature modulo p are in $\mathbb{F}_p(n^p - n)$. For each equation/operator there will be some finite number of such p that give us trouble and so we will use a few different primes p to better distinguish equations/operators. (We found that after three primes the extra computation time was not worth the extra resolution gained.)

6.2 Algorithm Step 2

It is necessary to check both the positive and negative square roots because, in general, there is no gauge transformation from L to $L \otimes (\tau + 1)$. The term product performed in this step is sufficient by Theorem 3.4.

6.3 Algorithm Step 3

We search the Sloane database for sequences satisfying second order, irreducible, homogeneous, recurrence equations and sequences satisfying third order equations that are the Least Common Left Multiple of such a second order equation and a first order equation. The results are separated into collections such that there exists a gt-transformation between any two members of a collection. One or two representatives of each collection are chosen depending on whether, after an appropriate gt-transformation, we have one or two linearly independent sequences in S (see Definition 2.6).

REFERENCES

[1] N. J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. World-Wide Web URL www.research.att.com/~njas/sequences/ 1, 6

[2] M. van Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. In *Journal of Pure and Applied Algebra*, **139**: 109-131, 1999. 1

[3] T. Cluzeau, M. van Hoeij. Computing Hypergeometric Solutions of Linear Recurrence Equations. In *Applicable Algebra in Engineering, Communication and Computing*, **17**(2): 83-115, 2006. 5.1

[4] R. Feng, M. F. Singer, M. Wu. Liouvillian solutions of difference-differential equations. To appear in *Journal of Symbolic Computation*, 2009. 5

[5] M. Petkovšek, H. S. Wilf, D. Zeilberger. $A = B$. With a foreword by Donald E. Knuth. A. K. Peters, Ltd., Wellesley, MA, 1996. 1

[6] M. van der Put, M. F. Singer. Galois theory of difference equations. In *Lecture Notes in Mathematics*, vol. **1666**, Springer, Berlin, 1997. 1, 1, 6.1

[7] P. A. Hendriks, M. F. Singer. Solving difference equations in finite terms. In *Journal of Symbolic Computation*, **27**: 239-259, 1999. 4.3, 5

[8] R. Vidunas. Contiguous relations of hypergeometric series. In *The Journal of Computational and Applied Mathematics*, vol. **153**: 507-519, 2003. 4.1.1, 4.1.1

[9] M. F. Singer. Testing reducibility of linear differential operators: a group theoretic perspective. In *Applicable Algebra in Engineering, Communication and Computing*, **7**(2): 77-104, 1996. 3

[10] S.A. Abramov. EG-Eliminations. In *Journal of Difference Equations and Applications*, vol. **5** 393-433, 1999. 3, 4

[11] G. Levy. Implementation. World-Wide Web URL www.math.fsu.edu/~glevy 1

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