Closed Form Solutions

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ISSAC 2017

[1] Supported by NSF 1618657

What is a closed form solution?

Example: Solve this equation for $y = y(x)$.

$$y' = \frac{4 - x^3}{(1-x)^2} e^x$$

Definition

A **closed form** solution is an expression for an **exact** solution given with a **finite amount of data**.

This is not a closed form solution:

$$y = 4x + 6x^2 + \frac{22}{3}x^3 + \frac{95}{12}x^4 + \cdots$$

because making it exact requires infinitely many terms.

The Risch algorithm finds a closed form solution:

$$y = \frac{2 + x^2}{1 - x} e^x$$

Risch algorithm (1969)

Previous slide: A **closed form** solution is an **expression** for an exact solution with only a finite amount of data.

Risch algorithm finds (if it exists) a **closed form** solution y for:

$$y' = f$$

To make that well-defined, specify **which expressions** are allowed:

Define E_{in} and E_{out} such that:

- Any $f \in E_{\text{in}}$ is allowed as input.
- Output: a solution iff $\exists$ solution $y \in E_{\text{out}}$.

Risch: $E_{\text{in}} = E_{\text{out}} = \{$**elementary functions**$\}$
$= \{$**expressions** with $\mathbb{C}(x)$ $\exp$ $\log$ $+$ $-$ $\cdot$ $\div$ composition and algebraic extensions$\}$.

Liouvillian solutions

Kovacic' algorithm (1986)

1. Solves homogeneous differential equations of order 2

$$a_2 y'' + a_1 y' + a_0 y = 0$$

   (Risch: inhomogeneous equations of order 1)

2. It finds solutions in a larger class:

$$E_{\mathrm{out}} = \{\text{Liouvillian functions}\} \supsetneq \{\text{elementary functions}\}$$

3. but it is more restrictive in the input:

$$a_0, a_1, a_2 \in \{\text{rational functions}\} \subsetneq \{\text{elementary functions}\}$$

Remark

$\exists$ common functions that are not Liouvillian.

Allow those as closed form $\rightsquigarrow$ need other solvers.

A non-Liouvillian example

Let

$$y \ := \ \oint_\gamma \exp\left(\frac{t^2+x+1}{x-t}\right)\frac{t}{x^2+x+1}\mathrm{d}t$$

Zeilberger's algorithm $\rightsquigarrow$ an equation for y:

$$(x^4-x)y'' + (4x^4+2x^3-3x^2-7x+1)y' + (6x^3-9x^2-12x+3)y = 0$$

Closed form solutions were thought to be rare.

But (for order 2) **telescoping equations** often (always?) have closed form solutions:

$$\exp(-2x)\cdot\left(I_0\left(2\sqrt{x^2+x+1}\right) - \frac{x+1}{\sqrt{x^2+x+1}}I_1\left(2\sqrt{x^2+x+1}\right)\right)$$

$$\exp(-2x)\cdot\left(K_0\left(2\sqrt{x^2+x+1}\right) + \frac{x+1}{\sqrt{x^2+x+1}}K_1\left(2\sqrt{x^2+x+1}\right)\right)$$

Globally bounded equations

So-called **globally bounded** equations are common in:

- combinatorics (Mishna's tutorial)
- physics (Ising model, Feynman diagrams, etc.)
- Period integrals, creative telescoping, diagonals.

Conjecture

Globally bounded equations (of order 2) have closed form solutions.

In other words: Closed form solutions are common.

Local to global strategy

Risch: Given elementary function f, solve:

$$y' = f$$

$\{\text{poles of } y\} \subseteq \{\text{poles of } f\} = \text{known}$

Kovacic: Given polynomials $a_0, a_1, a_2 \in \mathbb{C}[x]$, solve:

$$a_2 y'' + a_1 y' + a_0 y = 0$$

$\{\text{poles of } y\} \subseteq \{\text{roots of } a_2\} = \text{known}$

Local to global strategy

$\{$poles of $y\}$ + $\{$terms in polar parts$\}$ (+ other data) $\rightsquigarrow$ y

Local data: Classifying singularities

Example:

$$y = \exp(r) \quad \text{where} \quad r = \frac{1}{x^3} + \frac{5}{x^2} + \frac{3}{x-1} + 5 + 7x$$

y has **essential singularities** at the poles of r.

Definition

$y_1, y_2 \neq 0$ have an **equivalent singularity** at $x = p$ when y_1/y_2 is meromorphic at $x = p$.

Equivalence class of y at	$x = 0$,	$x = 1$,	$x = \infty$	(local data)
$\rightsquigarrow$ Polar part of r at	$x = 0$,	$x = 1$,	$x = \infty$	(local data)
$\rightsquigarrow$	$\frac{1}{x^3} + \frac{5}{x^2}$	$\frac{3}{x-1}$	$7x$	(local data)
$\rightsquigarrow$ r (up to a constant term)				(global data)
$\rightsquigarrow$ y (up to a constant factor)				(global data)

Reconstructing solutions from local data

Recall: y_1, y_2 have **equivalent singularity** at $x = p$
if y_1/y_2 is meromorphic at $x = p$.

Hence:

$$y_1, y_2 \text{ equivalent at every } p \in \mathbb{C} \bigcup \{\infty\}$$
$$\Longleftrightarrow$$
$$y_1/y_2 \text{ meromorphic at every } p \in \mathbb{C} \bigcup \{\infty\}$$
$$\Longleftrightarrow$$
$$y_1/y_2 \in \mathbb{C}(x)$$

Hence:

$$\{\text{Eq. class of } y \text{ at all } p\} \iff y \text{ up to a rational factor}$$

For a differential equation L **can compute**:

{**generalized exponents** of L at p} $\approx$ {Eq. classes of solutions}

Choose the right one at each p $\rightsquigarrow$ a solution (up to $\approx$)

Example: generalized exponents

Example: let L have singularities $\{0, 3, 4\}$, order 2, and solutions:

$$y_1 = (x^4 - 2x + 2) \cdot \exp(\int \frac{e_{0,1}}{x} + \frac{e_{3,1}}{x-3} + \frac{e_{4,1}}{x-4})$$

$$y_2 = (x^3 + 3x - 7) \cdot \exp(\int \frac{e_{0,2}}{x} + \frac{e_{3,2}}{x-3} + \frac{e_{4,2}}{x-4})$$

where $e_{p,i} \in \mathbb{C}[\,\frac{1}{x-p}\,]$ encodes the **polar part** $\frac{e_{p,i}}{x-p}$ at $x = p$.

These $e_{p,i}$ are the **generalized exponents** of L at $x = p$ and can be computed from L:

$$E_0 = \{e_{0,1},\ e_{0,2}\}, \quad E_3 = \{e_{3,1},\ e_{3,2}\}, \quad E_4 = \{e_{4,1},\ e_{4,2}\}$$

To find y_1 we need to **choose the correct element** of each E_p.

The example has $2^3 = 8$ combinations.
One combination $\rightsquigarrow y_1$, another $\rightsquigarrow y_2$, other six $\rightsquigarrow$ nothing.

Can reduce #combinations (e.g. Fuchs' relation)

Generalized exponents $\rightsquigarrow$ hyper-exponential solutions:

Let $a_0, \ldots, a_n \in \mathbb{C}[x]$ and $L(y) := a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0$.

Hyper-exponential solution: $y = \exp(\int r)$ for some $r \in \mathbb{C}(x)$.

$\{$**generalized exponent** of such y **at all singularities** p of $L\}$

$\rightsquigarrow$

y up to a **polynomial** factor (generalized exponent $\approx$ eq. class)

Algorithm hyper-exponential solutions:

1. *Compute generalized exponents* $\{e_{p,1}, \ldots, e_{p,n}\}$ *at each singularity* $p \in \mathbb{C} \bigcup \{\infty\}$ *of* L.
2. *For each* **combination** $e_p \in \{e_{p,1}, \ldots, e_{p,n}\}$ *(for all* p*) compute* **polynomial solutions** *of a related equation.*

Same strategy for difference equation

Combine **generalized exponents** $\rightsquigarrow$ hyper-exponential solutions.

To do the same for **difference equations** we need the difference analogue of generalized exponents:

Difference case: $p = \infty$ is similar to the differential case.

But a finite singularity is not an element $p \in \mathbb{C}$.

Instead it is an element of $\mathbb{C}/\mathbb{Z}$ because

$$y(x) \text{ singular at } p \iff y(x+1) \text{ singular at } p$$

is only true for $p = \infty$.

(1997): Generalized exponents

(1999): Difference case analogue:

generalized exponents at $p = \infty$ and

valuation growths at $p \in \mathbb{C}/\mathbb{Z}$

$\rightsquigarrow$ Algorithm for **hypergeometric solutions**.

Closed form solutions of linear differential equations:

Goal: define, then find, **closed form solutions** of:

$$a_n y^{(n)} + \cdots + a_1 y' + a_0 y = 0 \quad \text{with} \quad a_0, \ldots, a_n \in \mathbb{C}(x). \qquad (1)$$

The **order** is n (we assume $a_n \neq 0$).

Consider **closed form** expressions in terms of **functions** that are:

1. well known, and
2. **D-finite**: satisfies an equation of form (1).

D-finite of order 1 = hyper-exponential function.

Well known D-finite functions of order 2:

- Airy functions, Bessel functions, Kummer, Whittaker, ...
- Gauss hypergeometric function ${}_2F_1(a, b; c \mid x)$

  Klein's theorem: Liouvillian solutions are ${}_2F_1$ expressible.

Bessel type solutions:

Idea for constructing a Bessel-solver:

- Bessel functions have an essential singularity at $x = \infty$.
- Just like the function $\exp(x)$.
- So the strategy for hyper-exponential solutions may work for Bessel-type solutions as well.
- It also works for Airy, Kummer, Whittaker, and hypergeometric ${}_pF_q$ functions if $p + 1 \neq q$.

Later: Other strategies for Gauss hypergeometric ${}_2F_1$ function (to solve **globally bounded** equations of order 2).

Question: Which Bessel expressions should the solver look for?
Which Bessel expressions are D-finite?

Bessel type closed form expressions

Let $B_\nu(x)$ be one of the Bessel functions, with parameter ν.

Bessel type closed form expressions should allow:

- algebraic functions
- exp and log
- composition
- field operations
- differentiation and integration
- and of course $B_\nu(x)$.

Example: $B_0(\exp(x))$ is a **Bessel type closed form expression** but is **not relevant** for (1) since it is **not D-finite**.

Question: which Bessel type expressions are D-finite?

D-finite functions:

A function $y = y(x)$ is **D-finite of order** n if it satisfies a differential equation of order n with rational function coefficients.

Operations that don't increase the order:

1. $y(x) \mapsto y(f)$ for some $f \in \mathbb{C}(x)$ called **pullback function**.
2. $y \mapsto r_0 y + r_1 y' + \cdots + r_{n-1} y^{(n-1)}$ for some $r_i \in \mathbb{C}(x)$.
3. $y \mapsto \exp(\int r) \cdot y$ for some $r \in \mathbb{C}(x)$.

Operations that can increase the order:

4. Same as (1),(2),(3) but with algebraic functions f, r_i, r.
5. $y_1, y_2 \mapsto y_1 + y_2$ order n_1, n_2 $\rightsquigarrow$ order $\leq n_1 + n_2$
6. $y_1, y_2 \mapsto y_1 \cdot y_2$ order n_1, n_2 $\rightsquigarrow$ order $\leq n_1 \cdot n_2$

   Special case: $y \mapsto y^2$ order n $\rightsquigarrow$ order $\leq \frac{n(n+1)}{2}$

Have algorithms to recover any combination of: (2), (3), (5), and part of (6).

Let $B_\nu(x)$ be one of the Bessel functions.

$B_\nu(\sqrt{x})$ is D-finite of order 2. Transformations (1), (2), (3) $\rightsquigarrow$

$$\exp(\int r) \cdot \left(r_0 \cdot B_\nu(\sqrt{f}) + r_1 \cdot B_\nu(\sqrt{f})'\right) \qquad (2)$$

is D-finite of order 2 for any $r, r_0, r_1, f \in \mathbb{C}(x)$.

Theorem (Quan Yuan 2012)

Let k be a subfield of $\mathbb{C}$ and let L be a linear homogeneous differential equation over $k(x)$ of **order 2**.

If $\exists$ solution of form (2) with **algebraic functions** r, r_0, r_1, f *then $\exists$ solution with* **rational functions** $r, r_0, r_1, f \in k(x)$.

Bessel-type solutions of higher order equations:

$\rightsquigarrow$ Add transformations (4),(5),(6).

Finding Bessel type solutions

$a_2 y'' + a_1 y' + a_0 y = 0$ where $a_0, a_1, a_2 \in \mathbb{C}[x]$.

Goal: Find Bessel-type solutions.

Idea: Recover the pullback function f in transformation (1) from data that is **invariant** under transformations (2),(3).

Hyper-exponential solutions:

Generalized exponents $\rightsquigarrow$ $\{$polar parts of $f\}$ $\rightsquigarrow$ f

Bessel-type solutions:

Generalized exponents $\rightsquigarrow$ $\{\lceil$half$\rceil$ of terms of polar parts of $f\}$

$\rightsquigarrow$ need **more data** to find f.

More data: regular singularities $\rightsquigarrow$ roots of order $\notin \mathrm{denom}(\nu) \cdot \mathbb{Z}$

Combine data $\rightsquigarrow$ f **except in one case**: $\mathrm{denom}(\nu) = 2$ that "happens" to be solvable with Kovacic

Local to global strategy for difference equations

Use local data that is **invariant** under the difference analogue of transformations (2),(3):

- Giles Levy (Ph.D 2009)
- Yongjae Cha (Ph.D 2010)

Example: oeis.org/A000179 (Ménage numbers)

Recurrence operator:

$$(\tau + 1) \circ \left(n\tau^2 - (n^2 + 2n)\tau - n - 2\right)$$

where τ *is the* **shift-operator**.

$$\textbf{solver} \rightsquigarrow c_1 \cdot n \cdot I_n(-2) + c_2 \cdot n \cdot K_n(2) + c_3 \cdot \epsilon(n)$$

where $I_n(x)$ *and* $K_n(x)$ *are* **Bessel** *functions and* $\epsilon(n)$ *is a complicated expression that converges to 0 as* $n \to \infty$.

Result:

$$\mathrm{A000179}(n) = \mathrm{round}\left(\frac{2n}{e^2} \cdot K_n(2)\right) \quad (\mathrm{for}\ n > 0)$$

${}_2F_1$-type solutions

The **Gauss hypergeometric function** is:

$$ {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, x\right) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} x^n $$

where $(a)_n = a \cdot (a+1) \cdots (a+n-1)$.

If $L(y) = 0$ is a **globally bounded** equation of order 2 then it conjecturally has algebraic or ${}_2F_1$-type solutions:

$$ y = \exp\left(\int r\right) \cdot \left(r_0 \cdot {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, f\right) + r_1 \cdot {}_2F_1\left(\begin{matrix} a, b \\ c \end{matrix} \,\middle|\, f\right)' \right) $$

Problem: The local to global strategy:

$$ \text{invariant local data} \rightsquigarrow \text{pullback function } f \rightsquigarrow y $$

works for many functions, but ${}_2F_1$ can be problematic because f can be large even if the amount of local data is small.

${}_2F_1$ example

Small equation:

$$4x(x^2 - 34x + 1)y'' + (8x^2 - 204x + 4)y' + (x - 10)y = 0$$

The smallest solution:

$$\frac{\sqrt{3 - 3x - \sqrt{x^2 - 34x + 1}}}{x + 1} \cdot {}_2F_1\left(\begin{matrix}\frac{1}{3}, \frac{2}{3} \\ 1\end{matrix} \,\middle|\, f\right)$$

has

$$f = \frac{(x^3 + 30x^2 - 24x + 1) - (x^2 - 7x + 1)\sqrt{x^2 - 34x + 1}}{2(x + 1)^3}$$

How to construct f from a **small amount** of **invariant local data**:

- Exponent-differences: 0, 0, $\frac{1}{2}$ $\pmod{\mathbb{Z}}$
- at the singularities: $x = 0$, $x = \infty$, $x^2 - 34x + 1 = 0$

${}_2F_1$-type solutions and related topics:

- Tingting Fang (Ph.D 2012)
  - Compute D-module automorphisms $\rightsquigarrow$ descent.
    (also useful for non ${}_2F_1$ cases and for order > 2)
- Vijay Kunwar (Ph.D 2014)
  - Small f: Construct from invariant local data.
  - Large f: Build tables and use combinatorial objects (such as dessins d'enfant) to prove completeness.
- Erdal Imamoglu (Ph.D 2017)
  - If transformation (2) is not needed: quotient method.
  - Otherwise: Differential analogue of POLRED $\rightsquigarrow$ simpler equations. Then use quotient method.
- Wen Xu (Ph.D in progress)
  - Multivariate generalizations of ${}_2F_1$ such as Appell F_1.

POLRED: Cohen and Diaz Y Diaz (1991)

Algebraic computations often lead to an equation:

$$f(x) = 0$$

for some irreducible $f \in \mathbb{Q}[x]$. Such f defines a number field:

$$K = \mathbb{Q}[x]/(f)$$

In many computations there is no reason to assume that f is the simplest polynomial that defines K.

Algorithm POLRED

Input: Irreducible $f \in \mathbb{Q}[x]$.

Output: Monic $g \in \mathbb{Z}[x]$ for the same field:

$$K \cong \mathbb{Q}[x]/(g).$$

with near-optimal size for max(abs(coefficients of g)).

Differential analogue of POLRED

The following equation came from lattice path combinatorics
$\Longrightarrow$ **globally bounded**, conjecturally implies $\exists\ {}_2F_1$-type solutions

$$x(8x^2-1)(8x^2+1)(896x^5-512x^4+832x^3-127x^2-6x-12)\cdot y''$$
$$-(8x^2+1)(71680x^7-36864x^6+46080x^5-3528x^4-5280x^3$$
$$+155x^2+24x+36)\cdot y' + (1720320x^8-786432x^7+1078272x^6$$
$$-183360x^5+48384x^4-12464x^3-4560x^2-928x-96)\cdot y = 0$$

www.math.fsu.edu/∼eimamogl/hypergeometricsols

Finds smaller equation by imitating POLRED:

- Take the **differential module** for this equation.
- Compute its **integral elements**.
- Construct integral element Y with **minimal degree** at infinity.
- Equation for Y:

$$x(8x^2-1)(8x^2+1)\cdot Y'' + (320x^4-1)\cdot Y' + 192x^3\cdot Y = 0$$

Order > 2

If globally bounded equations of order 2 have ${}_2F_1$-type solutions, what about higher order?

Univariate generalization of ${}_2F_1$: hypergeometric ${}_pF_q$ functions.

Globally bounded order 3 equations need not be ${}_pF_q$-solvable.
Can construct a **univariate** example from **multivariate** hypergeometric functions (substitution $\rightsquigarrow$ **univariate**).

There are many multivariate hypergeometric functions.
A **particle zoo** of functions?

Fortunately, they have been organized in terms of polytopes:

A-**hypergeometric functions**
Gelfand, Kapranov, Zelevinsky (1990)
Beukers (ISSAC'2012 invited talk and recent papers)

Are globally bounded equations solvable in terms of such functions?

Order 3, Wen Xu (2017)

Trying to solve order >2 equations in terms of such functions leads to many questions, for instance: how they relate to each other? Do we need reducible A-hypergeometric systems?

Example: The Horn G_3 function satisfies a bivariate system of order 3. In the **reducible case** $a = 1 - 2b$ this function

$$G_3(1-2b, b \,|\, x, y)$$

satisfies the same bivariate differential equations as:

$$(1+3y)^{\frac{3}{2}b-1} y^{1-2b} \cdot {}_2F_1\left(\begin{matrix} \frac{1}{3} - \frac{b}{2}, \frac{2}{3} - \frac{b}{2} \\ \frac{1}{2} \end{matrix} \,\middle|\, \frac{(27xy^2 - 9y - 2)^2}{4(1+3y)^3} \right)$$

Found similar formulas for other reducible order 3 systems.

Thank you