AF:Small: A-Hypergeometric Solutions of Linear Differential Equations.

Project Description

Mark van Hoeij

September 1, 2016 — August 31, 2019

1 Introduction

In recent years the PI and his students have developed algorithms for finding closed form solutions of second order linear differential equations with rational function coefficients. The PhD thesis [33] of Quan Yuan gave provably complete algorithms to find closed form solutions in terms of many special functions, with one important exception: the Gauss hypergeometric ${}_2F_1$ function.

Since then, the PI and his students have developed several algorithms to find ${}_2F_1$-type solutions. Combined, nearly all ${}_2F_1$-cases are now covered. Equations with ${}_2F_1$-type solutions turned out to be remarkably common (e.g. [6, 9, 10, 24, 47]), leading to an unexpected conjecture:

Definition 1. *A linear homogeneous differential equation with polynomial coefficients is a* CIS-equation *if it has a CIS-solution: a non-zero solution that, possibly after scaling*[1]*, allows a* Convergent Integer power Series.

Conjecture 1. *Any second order CIS-equation is ${}_2F_1$-solvable.*

Solving such equations used to be difficult; closed form solutions can be complicated (e.g. http://oeis.org/A151329). With support from NSF 1319547, algorithms were developed that can quickly solve these equations. As a result, Conjecture 1 has now been tested on hundreds of examples coming from many sources.

[1] $y(x)$ is CIS, also called *globally bounded*, if it has an expansion $y(x) = \sum_{i=0}^{\infty} u_i (sx)^i$, with positive radius of convergence, for some $s \in \mathbb{C}$ and $u_i \in \mathbb{Z}$.

An integer sequence $\{u_n\}$ corresponds to an integer power series $\sum u_n x^n$, so it is not surprising that CIS-equations are common in combinatorics. But they are surprisingly common in physics as well, such as Calabi-Yau equations [38, 39, 46] or the Ising model [6, 10, 47].

A puzzling question remained. Why should second order CIS-equations from so many sources all be ${}_2F_1$-solvable, while the same sources also have 4th order CIS-equations that resist being solved by similar methods?

The hypergeometric ${}_2F_1$ function has many applications, and consequently, many generalizations have been introduced. The ${}_pF_q$-functions are univariate functions, but there are also multivariate hypergeometric functions such as Appell functions, Horn, Lauricella, etc. This ever growing zoo became organized when GKZ [52] introduced A-hypergeometric functions.

For a number of years it appeared as though Conjecture 1 does not extend to order > 3. The reason is because for solving univariate differential equations, the most natural functions to consider are univariate hypergeometric functions (the ${}_pF_q$-functions). The methods developed for ${}_2F_1$-type solutions generalize to ${}_pF_q$-type solutions (${}_4F_3$ for 4th order equations). However, of the hundreds of 4th order equations in the Calabi-Yau database [38], only a handful turned out to be ${}_4F_3$-solvable. Conjecture 2 below proposes the following answer to this puzzling situation: Conjecture 1 *does generalize* to higher order, but only if one allows *multivariate* hypergeometric functions (which become univariate via substitutions).

Conjecture 2. *Any CIS-equation can be solved in terms of A-hypergeometric functions.*

New methods will need to be developed that, given such an equation, select the right A-hypergeometric system and its parameters, and then find the correct substitutions.

In the GKZ system, the rank (= order) of an A-hypergeometric system corresponds to a polytope with normalized volume n, so one may additionally ask:

Question 1. *Are CIS-equations of order n solvable in terms of A-hypergeometric functions with polytopes of normalized volume $\leq n$?*

This leads to numerous other conjectures and questions, several of which can be tested with a finite computation. For example:

Question 2. *If an A-hypergeometric module of rank n is reducible, must its factors be solvable in terms of A-hypergeometric functions with rank $< n$?*

We checked this for one case, namely the Appell F_1 function, which is A-hypergeometric of rank 3. It is known (the so-called *resonant case* [52]) for which parameters an A-hypergeometric system becomes reducible. For each reducible Appell F_1 system, we computed the rank-2 factor and verified that it is ${}_2F_1$-solvable with the algorithms supported by NSF 1319547. These algorithms can solve non-trivial equations; this computation produced several formulas for the Appell F_1 function that are not known in the literature.

1.1 Goals

The main goal in this proposal is to develop algorithms to find solutions in terms of A-hypergeometric functions. Intermediate goals are:

1. Classify A-polytopes with normalized volume 3, then classify the corresponding A-hypergeometric systems.
2. Answer Question 2 for all resonant (i.e. reducible) A-hypergeometric systems of rank 3.
3. Develop algorithms that can solve in terms of A-hypergeometric functions of rank 3, starting with the Appell F_1 function. Then test Question 2 for rank-3 factors of higher-rank systems.
4. Classify A-polytopes of normalized volume 4, and develop algorithms to find solutions in terms of A-hypergeometric functions of rank 4.
5. Apply these algorithms to the Calabi-Yau database [38], which has over 400 CIS-equations, most of which have order 4. The algorithms to be developed in this proposal should allow us to either solve these equations, or find a counter-example to Conjecture 2.
6. Develop algorithmic tools to facilitate the above goals.

1.2 Relation to prior work

Christol's conjecture [61] states that a CIS function $y(x) = \sum u_i x^i$ satisfying a linear differential equation should be the *diagonal* of a multivariate rational function. This has recently been shown [45] to be equivalent with $\{u_n\}$ being a *multiple binomial sum*. Christol's conjecture differs from ours in an important way: it is not falsifiable with currently known methods.

We collected a list of CIS-equations from various sources (including hundreds from the Online Encyclopedia of Integer Sequences, oeis.org). There is no known method to decide if a CIS-equation comes from a diagonal, except if a diagonal or a multiple binomial sum is already known. So there is no way to know if the list has many, few, or no counter examples to Christol's conjecture.

In contrast, Conjecture 1 is falsifiable because for each explicit example we can test Conjecture 1 with the algorithms supported by NSF 1319547. Likewise, Conjecture 2 should become testable with the algorithms to be developed under this proposal.

In [50] Dwork conjectured that *globally nilpotent* [47] operators are solvable in terms of hypergeometric functions. This conjecture implies our conjectures because CIS implies globally nilpotent. However, a counter example to Dwork's conjecture was given by Krammer in [72]. It is not a counter example to Conjecture 1 since the function has no integer power series, even after scaling.

Many of the computational tools used for NSF 1319547 will need to be extended to higher order equations, and to multivariate systems. For example, the PI implemented DFactor in Maple, a program that can factor univariate differential operators. To test Question 2 properly, one needs to compute a factor of a multivariate system (we verified Question 2 for the Appell F_1 function by reducing to univariate equations via substitutions, and then reconstructing the bivariate result via interpolation, but continuing this approach would be tedious).

1.3 Broader Impacts of the Proposed Work

This project will provide valuable research experience for two graduate students, who will develop new algorithms that will significantly increase the capabilities of computers to solve differential equations. These algorithms will be made freely available.

Many branches of science have important impacts on society. Differential equations occur in almost every branch of science, and having closed form solutions is very useful in practical applications. Computer algebra systems are widely used and are of great value to society; they are an important part of the infrastructure for research and education. Within computer algebra, differential equations is one of the areas with the highest overall impact.

2 Notation and Examples

Let ∂ denote d/dx, so if $L = a_n\partial^n + \cdots + a_1\partial + a_0$ then $L(y)$ denotes $a_n y^{(n)} + \cdots + a_1 y' + a_0 y$. The Gauss hypergeometric function is defined as follows:

$$ {}_2F_1(a,b;c;\,x) = \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{x^n}{n!} \tag{1} $$

where the Pochhammer symbol $(a)_n$ is

$$ (a)_n = a(a+1)(a+2)\cdots(a+n-1) = \frac{\Gamma(n+a)}{\Gamma(a)}. $$

The ${}_2F_1$-function satisfies a second order equation $L(y) = 0$ where

$$ L = (x - x^2)\partial^2 + (c - (a+b+1)x)\partial - ab. \tag{2} $$

2.1 An example relating Conjecture 1 to Question 2.

The Apéry numbers

$$ u_n = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \tag{3} $$

have a generating function $y = \sum_{n=0}^{\infty} u_n x^n$ that satisfies the differential equation $L(y) = 0$ where

$$ L = (x^3 + 11x^2 - x)\partial^2 + (3x^2 + 22x - 1)\partial + (x+3) \tag{4} $$

L has a Convergent Integer power Series solution so it should be ${}_2F_1$-solvable according to Conjecture 1. The smallest ${}_2F_1$-solution is

$$ y \;=\; \frac{{}_2F_1\left(\frac{1}{12}, \frac{5}{12}; 1; f\right)}{\sqrt[4]{1 - 12x + 14x^2 + 12x^3 + x^4}} $$

where the *pullback function* is

$$ f = \frac{1728x^5(1 - 11x - x^2)}{(1 - 12x + 14x^2 + 12x^3 + x^4)^3} $$

Before the implementations supported by NSF 1319547 (see Section 6.3) it was hard to find such solutions. The main complication is that the factor in the denominator in the pullback function f is not visible in L. Note that the degree of f is 12 even though L has only 4 singular points (the roots of $x^3 + 11x^2 - x$ and the point at infinity).

The question in Conjecture 1 is not how to find such a solution, but why it should exist in the first place. Question 2 can help explain this as follows. The expression

$$y = \sum_{n,k} \binom{n}{k}^2 \binom{n+k}{k} x^n$$

can be written as an A-hypergeometric function by following the recipe from [44, Section 6], which produces the A-polytope and the parameters[2]. The polytope has normalized volume 6, and the system is resonant (reducible). The system has 7 variables $v_1, \ldots, v_7$. The function y is obtained by substituting $(v_1, \ldots, v_7) = (x, -1, 1, 1, 1, 1, 1)$. This substitution reduces the A-hypergeometric system of PDE's in 7 variables to a univariate differential operator of order 6:

$$\begin{aligned} L_6 &= x^4(x^2+11x-1)\partial^6 + x^3(23x^2+198x-13)\partial^5 \\ &\quad + x^2(171x^2+1081x-46)\partial^4 + 2x(245x^2+1024x-23)\partial^3 \\ &\quad + (506x^2+1142x-8)\partial^2 + (140x+100)\partial + 4. \end{aligned}$$

Since the system is resonant, L_6 must be reducible. The factorization can be obtained with Maple's DFactor (developed in the PI's PhD thesis). This way one recovers the second order operator L in (4) above. If Question 2 is true, then this would explain why Conjecture 1 is true for equations coming from binomial sums. This is progress because Question 2 is partially testable at the moment, and much more so by the end of the proposed project.

There are two versions of Question 2:

(i) Systems that become reducible by selecting resonant parameters.

(ii) Systems that become reducible by a substitution of the variables: $(v_1, \ldots, v_N) = (f_1, \ldots, f_N)$.

The answer to Question 2 may well depend on which version one considers.

Version (i) has already helped our implementation as follows. Take an A-hypergeometric system, select parameters for which it has a second order factor, and substitute arbitrary rational functions for the variables $v_1, \ldots, v_N$. The resulting second order equation should be ${}_2F_1$-solvable. No counter examples were found, but this did turn out to be an excellent way to find bugs in the implementation.

[2]This recipe can likely be used to give another proof to the Wilf-Zeilberger Conjecture [55, 60] and an alternative to Zeilberger's algorithm [79].

One way to find a (partial) proof for Question 2 is by comparing the monodromy (which can be often computed due to recent work of Beukers [43]) of resonant rank-n systems with that of rank $< n$ systems. However, our primary focus will be on developing algorithms, because these will be useful for many other applications.

2.2 An example of order 3

A univariate generalization of the Gauss hypergeometric function is the ${}_pF_q$ function

$$ {}_pF_q(a_1,\ldots,a_p;\, b_1,\ldots,b_q;\, x) = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{x^n}{n!} $$

The Calabi-Yau database [38] contains a large number of CIS-equations. The third order equations in this database turn out to be ${}_3F_2$-solvable (and ${}_2F_1$-solvable via the reduction of order from [69]). However, most of the hundreds of 4th order equations in the database are not ${}_4F_3$-solvable. So Conjecture 2 fails for order 4 if we restrict to univariate hypergeometric functions.

One of the many multivariate generalizations is Appell's F_1 function.

$$ F_1(a,b_1,b_2,c;\, x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(a)_{m+n}(b_1)_m(b_2)_n}{(c)_{m+n} m! n!} x^m y^n $$

We can select parameters a, b_1, b_2, c for which F_1 becomes CIS, meaning that $F_1(a,b_1,b_2,c;\, s_1x, s_2y)$ will have integer coefficients for suitable scaling factors $s_1, s_2 \neq 0$. If we substitute $(x,y) \mapsto (f_1(x), f_2(x))$ for some CIS-functions with $f_1(0) = f_2(0) = 0$ then we obtain a univariate CIS function, which will satisfy a third order operator since the F_1 system has rank 3.

The PI and graduate student Wen Xu took such an example

$$ F_1\left(\frac{1}{6}, \frac{1}{5}, \frac{1}{5}, 1;\, \frac{x+\sqrt{x^2+4x}}{2}, \frac{x-\sqrt{x^2+4x}}{2}\right) \tag{5} $$

and computed its minimal operator $L =$

$$ \begin{aligned} &900x^2(x+4)(43x-20)(2x-1)\partial^3 + 60x(5891x^3+9388x^2-11890x+3000)\partial^2 \\ &\quad +(235296x^3+30775x^2-191300x+36000)\partial + (3096x^2-5005x-1900). \end{aligned} $$

The symmetry $b_1 = b_2$ allowed the square-root in f_1, f_2 to disappear. We picked the parameters $1/6, 1/5, 1/5, 1$ in (5) in such a way that we can prove that L is not ${}_3F_2$-solvable[3]. The function (5) has an integer power series $\sum u_i x^i$ after scaling $x \mapsto x \cdot 2^4 3^3 5^2$ so L is CIS.

In this example L is F_1-solvable by construction. Conjecture 2 says that all CIS-equations should come from A-hypergeometric functions in this way.

3 Proving that a recurrence produces an integer sequence

Question 3. *Is there a method that, for a sequence given by a recurrence, can decide if the sequence is an integer sequence or not?*

Consider the sequence $\{u_n\}$ given by $u_0 = 1$, $u_1 = 3$, and

$$u_n = \frac{(11n^2 - 11n + 3)u_{n-1} + (n-1)^2 u_{n-2}}{n^2} \tag{6}$$

One finds $u_2 = 76/2^2 = 19$, $u_3 = 1323/3^3 = 147$, etc. The question is, how to prove that every u_n is an integer, despite the divisions by n^2? There is a short proof for this example, as follows: Show that u_n is the same as (3) by showing (e.g. with Zeilberger's algorithm [79]) that (3) satisfies recurrence (6). But what if no clearly-integer expression such as (3) is known?

Suppose for example $v_0 = 1$, $v_1 = 828$, and

$$v_n = 4\frac{(-592n^2 + 1184n - 385)v_{n-1} - 28^3(4n-5)(4n-11)v_{n-2}}{n^2} \tag{7}$$

Again $v_0, v_1, v_2, \ldots$ appear to be integers, but this time we have no formula such as (3) that quickly proves this. The relation to this project is that at the moment, the only method to prove that $v_n \in \mathbb{Z}$ for all n is by solving a differential equation.

The recurrence for v_n can (e.g. with Maple's gfun package) be converted to a differential equation for $y = \sum v_n x^n$

$$\left(\partial^2 + \frac{1}{x}\partial - \frac{36(28^3 \cdot x + 23)}{2^{12}7^3 \cdot x^3 + 2^6 37 \cdot x^2 + x}\right)(y) = 0 \tag{8}$$

This equation appears to be CIS because $y = 1 + 828x + \cdots$ appears to have integer coefficients. No current computer algebra system can find a

[3] All order-3 CIS equations we encountered in the literature and oeis.org turned out to be ${}_3F_2$-solvable; this example is the first that showed that Conjecture 2 restricted to univariate hypergeometric functions also fails for order 3.

${}_2F_1$-type solution for this equation, for that, we need the algorithms ([14] or [25]) developed with support of NSF 1319547. The resulting ${}_2F_1$-type expression is:

$$y \;=\; r_0 \cdot {}_2F_1\left(\frac{1}{12}, \frac{7}{12}; 1; f\right) + r_1 \cdot {}_2F_1\left(\frac{1}{12}, \frac{19}{12}; 1; f\right) \tag{9}$$

where the pullback function is:

$$f = \frac{-12^3 x(112^3 x^2 + 2368x + 1)}{(1 + 1152x)^2}$$

and

$$r_0 = \frac{2 + 2496x}{9(1 + 1152x)^{7/6}}, \qquad r_1 = \frac{7(1 + 1344x)^2}{9(1 + 1152x)^{7/6}}.$$

Such a solution can be used to prove[4] that (8) is indeed CIS and that $\{v_n\}$ is an integer sequence.

Question 3 is closely related to solving differential equations. At the moment, the only way to prove that $\{v_n\}$ is an integer sequence is by solving (8). A key motivation for this proposal is to answer Question 3 when the differential equation has order > 2. As shown in Section 2.1, this will require multivariate hypergeometric functions.

4 Tools needed for this project

The key step towards finding solutions such as (9) from Section 3 is to find the parameters and the pullback function f. This data is obtained by comparing exponent-differences at the singularities of the input equation (equation (8) in the example) with that of the Gauss hypergeometric ${}_2F_1$ equation (2). The pullback function f maps each of the so-called non-removable singularities of the input equation (8) to one of the three singularities $0, 1, \infty$ of (2).

This becomes considerably more technical for A-hypergeometric solutions. Take the Appell F_1 system for example. The singularities are now not points, but lines $\{x = 0, x = 1, x = \infty, x = y, y = 0, y = 1, y = \infty\}$ and there are two pullback functions f_1, f_2 instead of one. A non-removable

[4] Sums, products, and compositions, of CIS-functions are CIS, so equation (9) immediately implies that y is CIS. This in turn implies that there are only finitely many primes p that could appear in a denominator in the sequence $\{v_n\}$. So (9) reduces the proof to checking a finite set of primes.

singularity of the input equation L need no longer come from one, it could also come from multiple singularities of the Appell F_1 system.

The local asymtotic behavior of the ${}_2F_1$ equation (2) is determined by the local exponents. This too is more complicated for the F_1 system. To find the pullback functions f_1, f_2 one needs a precise relation between the local asymptotic behavior of the F_1 system, and the exponents of the input equation L.

Graduate student Wen Xu is currently studying these issues. She will then use the obtained formulas to develop algorithms to find f_1, f_2. Initially these algorithms will only cover low-degree pullback functions, just like [14]. To find high degree pullback functions, one could consider extending one of the approaches described in item 1 in Section 6.3, or perhaps, search for a new approach.

4.1 Reducing equations

The integer sequence listed at oeis.org/A151329 is the number of walks in $\mathbb{N}^2$, starting at (0,0), and consisting of n steps taken from $\{(-1,-1),(-1,1),(-1,0),(0,1),(1,-1),(1,0),(1,1)\}$. The generating function $y = \sum u_n x^n$ satisfies a 5th order differential operator, which factors as a product of a second order factor L_2 and three first order factors. The explicit expression for the generating function given at oeis.org/A151329 was obtained by computing the ${}_2F_1$-type solutions of L_2, and applying three integrals that correspond to the first order factors. The operator L_2 is a large expression. It has a solution of a form similar to (9), however, this time r_0, r_1 are much larger. The solution can not be written with a single ${}_2F_1$ term, otherwise, the implementation [12] of graduate student Erdal Imamoglu (supported by NSF 1319547) would have found it.

The algorithm in [12] motivates the following question: Given a large equation L_2, how to find a so-called gauge-transformation that will reduce L_2 to an easier equation $\tilde{L}$, one where the solution can be written using a single ${}_2F_1$ term? That equation will then be solvable with [12]. Solving it, and applying the inverse gauge transformation $Y \mapsto r_0 Y + r_1 Y'$ will then produce the solutions of L_2.

Erdal Imamoglu has recently implemented a method that is very effective at finding this gauge-transformation. So far, it appears that his implementation [25] always manages to reduce to an equation whose solution involves only one ${}_2F_1$ term. This observation should be investigated in more details. Currently only order 2 is implemented, but if this strategy also works for higher order, it could simplify the algorithms significantly. For an equation of order n, it would mean that instead of searching for a solution with n terms $r_0 Y + r_1 Y' + \cdots r_{n-1} Y^{(n-1)}$, it suffices to search for a solution with one term. Such a reduction would allow us to focus on the problem of finding the parameters and pullback function(s) for the A-hypergeometric function.

Differential equations and recurrence relations can be converted to one another. The PI and YongJae Cha, a former student, have developed an algorithm [15] to find gauge-transformations for recurrence relations. Investigating how this interacts with reducing differential equations would be interesting, particularly for Question 3.

5 Liouvillian solutions

The celebrated Kovacic algorithm [71] can compute all Liouvillian solutions of second order equations and is implemented in several computer algebra systems. Liouvillian solutions are solutions that one can write in terms of exponentials, logarithms, integration symbols, algebraic extensions, and combinations thereof. In particular, they can be expressed without hypergeometric functions.

Klein's theorem provides an alternative representation. It says that for irreducible second order equations, Liouvillian solutions can be expressed with the hypergeometric ${}_2F_1$ function instead of algebraic extensions. An algorithm to find Liouvillian solutions in this form was developed by J.A. Weil and the PI [70]. It turned out that expressing Liouvillian solutions in ${}_2F_1$ form is much more efficient and leads to much smaller[5] expressions. For this reason, this algorithm is now the default in Maple. In fact, even if one wants Liouvillian solutions in Kovacic form, the most efficient method is to first compute them in ${}_2F_1$-form with [70] and then convert.

There are algorithms for computing Liouvillian solutions of higher order equations [62, 63]. However, only special cases (e.g. imprimitive differential Galois group) have been incorporated in a computer algebra system because for most Galois groups, the solutions become impractically large. These Liouvillian solutions should be expressible in terms of A-hypergeometric functions, with much smaller expression sizes.

[5] www.math.fsu.edu/~hoeij/papers/issac05/6.pdf gives an example with a more than 100-fold reduction in expression size

6 Results from prior NSF support.

During 2010–2015, the PI published 11 journal papers [1-11], 12 conference papers [12-23], 8 preprints [24-31]. Four of the PI's Ph.D students completed their thesis [32-35] during this time, and the PI co-advised for two more theses. The PI is currently supported by NSF 1319547, which also supports two RA's and one student from Brazil visiting for a year.

Title: AF:Small: Linear Differential Equations with a Convergent Integer Series Solution. NSF 1319547, 09/01/13 – 08/31/16, $479,405.

The Project Description of NSF 1319547 (as well as NSF 1017880 and 0728853) can be viewed at www.math.fsu.edu/∼hoeij/papers.html

6.1 Broader Impacts (NSF 1319547)

The PI and two graduate students will develop algorithms that will significantly increase the capabilities of computers to solve differential equations. This in turn will be very valuable to the many parts of science and engineering that use differential equations. Techniques from number theory and algebraic geometry will be used to tackle the theoretical goals.

The benefit to society is manifold but indirect; computer algorithms do not build bridges, but they are useful for designing bridges, studying ocean waves, fiber optics, quantum mechanics, population dynamics, etc., the list of applications of differential equations is long and diverse.

6.2 Intellectual merit (NSF 1319547)

The proposed work builds on recent work of the PI and his graduate students to develop algorithms for solving linear differential equations. A practical goal is to extend these algorithms to solve every CIS-equation of order < 4.

The main theoretical goal is to prove completeness for the proposed algorithms, in the sense that they either construct a closed form solution or a proof that such solutions do not exist. To accomplish this goal, the PI will use results from modular curves, number theory, Belyi maps and dessins d'enfants.

The PI's approach is very effective for CIS-equations of order < 4, which raises the theoretical question if all such equations have closed form solutions. The PI will study this question and compare with related concepts, such as G-functions, globally nilpotent operators, and Calabi-Yau operators.

6.3 Overview of results from NSF 1319547

1. **Linear differential equations**:

   Not long ago it was thought that closed form solutions are rare. However, for every integer sequence in oeis.org it turned out that if its generating function is convergent and satisfies a second order linear differential equation, then that equation is solvable in closed form. We can now solve these equations automatically with the algorithms supported by NSF 1319547. These algorithms are being used by other researchers to find closed form expressions in applications such as physics and combinatorics.

   Our first approach was to divide the problem of finding ${}_2F_1$ type solutions in two categories; the "easy" cases where f has low degree [14], or where the roots and poles of f can be read from the singularities. The hard cases are when f is an algebraic function, see item 2 below, or, where f has many roots or poles that do not appear among the singular points, as illustrated in (4) in Section 2.1. This situation can only occur when f has a very specific branching pattern. Such f correspond, up to Mobius-equivalence, to combinatorial objects called dessins d'enfants and near-dessins.

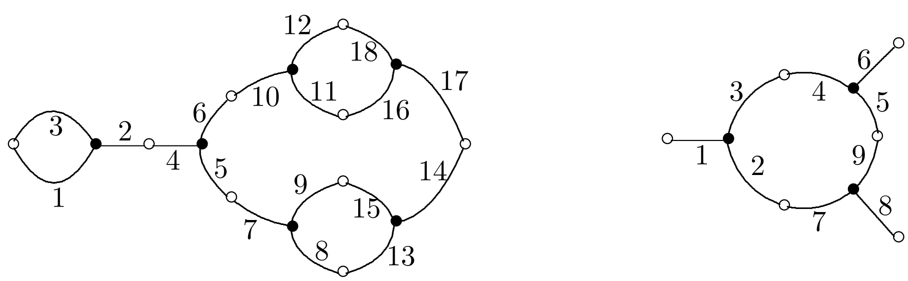

Figure 1: two dessins d'enfants

These are combinatorial objects. So for equations with a fixed number of singular points, it should in principle be possible to tabulate them with a finite computation:

- A Heun equation is a differential equation with $d = 4$ singular points. The PI and Raimundas Vidunas determined a table of all rational Belyi maps that can occur as a pullback function between a Heun and a hypergeometric equation. The result is the largest table in the literature of Belyi functions and dessins d'enfants. To find them it was necessary to develop numerous new algorithms. The table and algorithms are available at [1]. The table gives all (up to easy-to-recover transformations) Heun equations that are ${}_2F_1$-solvable with a rational pullback.

- With Vijay Kunwar (Ph.D 2014) this work has been extended to CIS-equations with $d = 5$ singular points. The tables, algorithms and preprint are available [26]. The emphasis in this work is to prove that the tables are complete, in order to ensure that the resulting differential solver will be complete whenever the equation has $d = 4$ or $d = 5$ non-removable singularities. For $d = 5$ we have to catalogue not only dessins d'enfants, but also near-dessins, which are braid orbits of 4-constellations (lists of 4 permutations). We have developed algorithms to find all near-dessins for $d = 5$ and have computed the corresponding pullback functions f, which in this context are one-dimensional families. Combining this work with [19, 14, 32, 26] we now have complete algorithms for large classes of equations.
- The tables for $d = 4$ and $d = 5$ are very large, extending them to > 5 singularities would not be practical. An algorithm [12] for any d has been developed with graduate student Erdal Imamoglu. Though completeness is not (yet) proven, the algorithm is very effective in practice, especially after the recent addition of the integral basis method [25].

  In summary, we now have provably[6] complete algorithms for large classes of second order equations, and a very effective algorithm for the remaining second order equations. So the focus in the proposed project will be on higher order equations. A lot of new tools will need to be developed for this, because the current tools are for the univariate case, which will not suffice for order > 2 as shown in Section 2.2.

2. **Modular curves**: The work with Vidunas and Kunwar classified rational functions that can occur as pullback functions for ${}_2F_1$-type solutions, but occasionally, an algebraic function is needed as well (see oeis.org/A005259 for an example where a square-root is needed inside the hypergeometric function). These algebraic cases correspond to modular curves $X_0(N)$. To tabulate them, the PI has written a website www.math.fsu.edu/~hoeij/files/X0N with data about $X_0(N)$ in

[6] This could be important in the event of a candidate counter-example to Conjecture 1.

computer readable format. It also contains an algorithm PuiseuxX0N and a corresponding preprint.

The PI has published a joint paper [2] with Maarten Derickx on the gonality of the modular curve $X_1(N)$, and written a preprint [27] with Derickx and Zeng. The websites with the data for these papers are in the folders X1N and XH under www.math.fsu.edu/∼hoeij/files

The PI has found a method [31] to find points on the modular curve $X_1(N)$ that are defined over minimal number fields. There is an ongoing project with Maarten Derickx to prove that the resulting list of degrees is complete for $N \leq 40$.

3. **Subfields**: The PI's algorithm [3] for computing subfields of number fields has been incorporated into Magma. The number of subfields (denoted m) is not polynomially bounded, and the complexity of computing all subfields has a term that depends on m. In joint work with Jonas Szutkoski, a visiting student from Brazil, we improved the complexity by minimizing this term. This improvement can be observed in Jonas' implementation; it outperforms Magma when m is large.

   Both [3] and the new algorithm compute a set $L_1, \ldots, L_r$ of so-called generating subfields. Then [3] computes all subfields by computing intersections of $L_1, \ldots, L_r$ with linear algebra. The key idea of the new algorithm is that this step can be done much faster as follows: One can associate a partition of $\{1, \ldots, r\}$ to each L_i, in such a way that intersecting subfields corresponds to a quick operation on these partitions. This way the entire subfield lattice can be obtained from $L_1, \ldots, L_r$ in $\mathcal{O}(r^{5/2}m)$ bit operations.

4. **Integral basis**: The PI and Mike Stillman have developed a new algorithm for computing an integral basis in an algebraic function field [28]. One application of integral basis is, for an algebraic function field given by a complicated defining polynomial, to find a smaller polynomial defining an isomorphic function field. Remarkably, a similar approach turned out to be highly effective for differential equations as well, making Erdal's implementation of the quotient method [12] far more powerful [25].

References

[1] M. van Hoeij and R. Vidunas, *Belyi functions for hyperbolic hypergeometric-to-Heun transformations*, Journal of Algebra, **441**, p. 609-659 (2015). Data at: www.math.fsu.edu/∼hoeij/Heun

[2] M. Derickx and M. van Hoeij, *Gonality of the modular curve* $X_1(N)$, Journal of Algebra, **417**, p. 52-71 (2014). Accompanying data at: www.math.fsu.edu/∼hoeij/files/X1N

[3] M. van Hoeij, J. Klüners, and A. Novocin, *Generating Subfields*, J. of Symbolic Computation, **52**, p. 17-34, (2013).

[4] J. Fullwood and M. van Hoeij, *On stringy invariants of GUT vacua*, Communications for Number Theory in Physics, **07** No. 4, p. 551-579 (2013)

[5] M. van Hoeij and A. Novocin, *Gradual Sub-lattice Reduction and a New Complexity for Factoring Polynomials*. Algorithmica 63(3): 616-633, (2012).

[6] M. Assis, S. Boukraa, S. Hassani, M. van Hoeij, J-M. Maillard, B.M. McCoy, *Diagonal Ising susceptibility: elliptic integrals, modular forms and Calabi-Yau equations*, J. Phys. A: Math. Theor. **45** 075205 (2012). In the: **Highlights of 2012 collection**.

[7] M. van Hoeij and V. Pal, *Isomorphisms of Algebraic Number Fields*, JNTB **24**, No. 2, p. 293-305, (2012).

[8] S. Abramov, M. Barkatou, M. van Hoeij, and M. Petkovšek, *Subanalytic Solutions of Linear Difference Equations and Multidimensional Hypergeometric Sequences*, J. Symbolic Comput. **46**, p. 1205-1228, (2011).

[9] A. Bostan, F. Chyzak, M. van Hoeij, L. Pech, *Explicit formula for the generating series of diagonal 3D rook paths*, Seminaire Lotharingien de Combinatoire, B66a (2011).

[10] A. Bostan, S. Boukraa, S. Hassani, M. van Hoeij, J.-M. Maillard, J-A. Weil and N. Zenine, *The Ising model: from elliptic curves to modular forms and Calabi-Yau equations*, J. Phys. A: Math. Theor. **44** 045204 (2011).

[11] A. Bostan, M. Kauers, with an appendix by M. van Hoeij, *The Complete Generating Function for Gessel Walks is Algebraic*, Proc. Amer. Math. Soc. **138** p. 3063-3078, (2010).

[12] E. Imamoglu and M. van Hoeij, *Computing Hypergeometric Solutions of Second Order Linear Differential Equations using Quotients of Formal Solutions*, ISSAC'2015 Proceedings, p. 235-242 (2015).
Implementation at: www.math.fsu.edu/∼eimamogl/find_2f1

[13] M. van Hoeij and R. Vidunas, *Computation of Genus 0 Belyi Functions*, LNCS **8592**, p. 92-98 (2014).

[14] V. Kunwar and M. van Hoeij, *Second Order Differential Equations with Hypergeometric Solutions of Degree Three*, ISSAC'2013 Proceedings, p. 235-242 (2013).
Implementation at: www.math.fsu.edu/∼vkunwar/hypergeomdeg3

[15] Y. Cha and M. van Hoeij, *Rational elements of the tensor product of solutions of difference operators*, Proceedings of the Tenth Asian Symposium on Computer Mathematics (2012).
Implementation at: sites.google.com/site/yongjaecha/code

[16] J. Fullwood and M. van Hoeij, *On Hirzebruch invariants of elliptic fibrations*, Proceedings of Symposia in Pure Mathematics, **85**, p. 355-366, (2012).
Implementation at: www.math.fsu.edu/∼hoeij/files/Hirzebruch

[17] M. van Hoeij, J. Klüners, and A. Novocin, *Generating Subfields*, ISSAC'2011 Proceedings, p. 345-352, (2011).

[18] W. Hart, M. van Hoeij, A. Novocin, *Practical Polynomial Factoring in Polynomial Time*, ISSAC'2011 Proceedings, p. 163-170, (2011).

[19] T. Fang and M. van Hoeij, *2-descent for Second Order Linear Differential Equations*, ISSAC'2011 Proceedings, p. 107-114, (2011).

[20] M. van Hoeij and Q. Yuan, *Finding all Bessel type solutions for Linear Differential Equations with Rational Function Coefficients*, ISSAC'2010 Proceedings, p. 37-44, (2010).

[21] M. van Hoeij and G. Levy, *Liouvillian Solutions of Irreducible Second Order Linear Difference Equations*, ISSAC'2010 Proceedings, p. 297-302, (2010).

[22] Y. Cha, M. van Hoeij, and G. Levy, *Solving Recurrence Relations using Local Invariants*, ISSAC'2010 Proceedings, p. 303-310, (2010).

[23] M. van Hoeij and A. Novocin, *Gradual sub-lattice reduction and a new complexity for factoring polynomials*, LATIN 2010, p. 539-553, (2010).

[24] A. Bostan, F. Chyzak, M. Kauers, L. Pech, and M. van Hoeij *Explicit Differentiably Finite Generating Functions of Walks with Small Steps in the Quarter Plane*, preprint,
slides at: specfun.inria.fr/seminar/20150427-FredericChyzak.pdf
2F1 expressions at: www.math.fsu.edu/∼hoeij/files/walks

[25] E. Imamoglu and M. van Hoeij, *Computing Hypergeometric Solutions of Second Order Linear Differential Equations using Quotients of Formal Solutions and Integral Bases*, preprint, submitted to JSC, implementation at: www.math.fsu.edu/∼eimamogl/hypergeometricsols

[26] M. van Hoeij and V. Kunwar, *Classifying (near)-Belyi maps with Five Exceptional Points*, preprint and accompanying data, www.math.fsu.edu/∼hoeij/FiveSing (2015).

[27] M. Derickx, M. van Hoeij, and Jinxiang Zeng. *Computing Galois representations and equations for modular curves $X_H(l)$*, submitted to J. Algebra.

[28] M. van Hoeij and M. Stillman, *Computing an Integral Basis for an Algebraic Function Field*, preprint, slides available at www.math.fsu.edu/∼hoeij/papers.html (2015).

[29] M. van Hoeij, *The complexity of factoring univariate polynomials over the rationals*, ISSAC'2013 tutorial (2013).
available at www.math.fsu.edu/∼hoeij/papers.html

[30] M. van Hoeij, *Computing Puiseux Expansions at Cusps of the Modular Curve $X_0(N)$*, arXiv:1307.1627v1 (2013).

[31] M. van Hoeij, *Low Degree Places on the Modular Curve $X_1(N)$*, arXiv:1202.4355 (2012–2014).
Accompanying data at: www.math.fsu.edu/∼hoeij/files/X1N

[32] V. Kunwar, *Hypergeometric Solutions of Linear Differential Equations with Rational Function Coefficients*, Ph.D thesis, implementations at www.math.fsu.edu/∼vkunwar (2014).

[33] Q. Yuan, *Finding all Bessel type solutions for Linear Differential Equations with Rational Function Coefficients*, Ph.D thesis, (2012). Implementation for finding all Airy/Bessel/Kummer/Whittaker type solutions is available at www.math.fsu.edu/∼qyuan

[34] T. Fang, *Solving Linear Differential Equations in Terms of Hypergeometric Functions by 2-Descent*, Ph.D thesis, (2012).

[35] Y. Cha, *Closed Form Solutions of Linear Difference Equations*, Ph.D thesis, (2010). implementation available at sites.google.com/site/yongjaecha

[36] G. Levy, *Solutions of Second Order Recurrence Relations*, Ph.D thesis, (2009). implementation available at www.math.fsu.edu/∼hoeij/glevy

[37] A. Novocin, *Factoring Univariate Polynomials over the Rationals*, Ph.D thesis, (2008).

[38] G. Almkvist, C. van Enckevort, D. van Straten, and W. Zudilin, *Tables of Calabi-Yau equations*, preprint: arxiv.org/abs/math/0507430 and database: www.mathematik.uni-mainz.de/CYequations/db/

[39] G. Almkvist, *The art of finding Calabi-Yau differential equations*, arxiv.org/abs/0902.4786 (2009).

[40] F. Beukers, *E-functions and G-functions*, lecture notes available at swc.math.arizona.edu/aws/2008/08BeukersNotesDraft.pdf (2008).

[41] F.Beukers, *Algebraic A-hypergeometric functions*, Invent. Math. **180** p. 589-610 (2010).

[42] F.Beukers, *Irreducibility of A-hypergeometric systems*, Indag. Math. **21** p. 30-39 (2011).

[43] F. Beukers, *Monodromy of A-hypergeometric functions*, J. reine angew. Math. DOI 10.1515/crelle-2014-0054 (2014).

[44] F. Beukers, *Hypergeometric Functions: how special are they?*, Notices of the AMS **61** p. 48-56 (2014).

[45] A. Bostan, P. Lairez, and B. Salvy, *Multiple binomial sums*, preprint arXiv:1510.07487 (2015).

[46] M. Bogner, *On differential operators of Calabi-Yau type*, Available at: ubm.opus.hbz-nrw.de/volltexte/2012/3191/pdf/doc.pdf (2012).

[47] A. Bostan, S. Boukraa, S. Hassani, J.-M. Maillard, J.-A. Weil, N. Zenine, *Globally nilpotent differential operators and the square Ising model*, J. Phys. A: Math. Theor. **42**, No. 12, (2009).

[48] A. Bostan, M. Kauers, *Automatic Classification of Restricted Lattice Walks*, Proceedings of FPSAC09, Hagenberg, Austria, p. 203-217, (2009).

[49] M. Bousquet-Mélou, M. Mishna, *Walks with small steps in the quarter plane*, Contemp. Math. **520**, p. 1-40, (2010)

[50] B. Dwork, *Differential operators with nilpotent p-curvature*, Am. J. Math. **112**, 749-786 (1990).

[51] S. Garoufalidis, *G-functions and multisum versus holonomic sequences*, arxiv.org/abs/0708.4354 (2007).

[52] I.M.Gelfand, M.M.Kapranov, A.V.Zelevinsky, *Generalized Euler integrals and A-hypergeometric functions*, Adv. in Math **84**, p. 255-271 (1990).

[53] R. Debeerst, M. van Hoeij, W. Koepf, *Solving Differential Equations in Terms of Bessel Functions*, ISSAC'08 Proceedings, 39-46, (2008).

[54] K. Belabas, J. Klüners, M. van Hoeij, and A. Steel *Factoring polynomials over global fields*, Journal de Théorie des Nombres de Bordeaux, **21**, 15-39 (2009).

[55] S.A. Abramov, *Applicability of Zeilberger's Algorithm to Hypergeometric Terms.* ISSAC'2002 Proceedings, p. 1-7. (2002).

[56] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, (1972).

[57] G. Andrews, R. Askey and R. Roy, *Special Functions*, Encyclopedia of Mathematics and its Applications, **71**, Cambridge University Press, (1999).

[58] K. Belabas, *A relative van Hoeij algorithm over number fields*, J. Symbolic Computation, **37**, 641-668 (2004).

[59] M. Berry, *Why are special functions special?* Physics Today, 54, no.4, 11-12, (2001). http://www.physicstoday.com/pt/vol-54/iss-4/p11.html

[60] S. Chen and C. Koutschan, *Proof of the Wilf-Zeilberger Conjecture*. preprint www.ricam.oeaw.ac.at/publications/reports/15/rep15-15.pdf (2015).

[61] G. Christol, *Globally bounded solutions of differential equations*. Analytic number theory **1434**. Lecture Notes in Math. Berlin: Springer, p. 45-64, (1990).

[62] M. van Hoeij, J.F. Ragot, F. Ulmer and J.A. Weil, *Liouvillian solutions of linear differential equations of order three and higher*. J. Symbolic Computation, **28**, 589-609 (1999). Implementation: www.math.fsu.edu/∼hoeij/files/impr_order3 (2004).

[63] O. Cormier, *On Liouvillian Solutions of Linear Differential Equations of Order 4 and 5*, ISSAC 2001 Proceedings, p. 93-100 (2001.

[64] M. van Hoeij, *Factoring polynomials and the knapsack problem*, J. of Number Theory, 95, 167-189, (2002).

[65] M.F. Singer, *Solving Homogeneous Linear Differential Equations in Terms of Second Order Linear Differential Equations*, American J. of Math., **107**, 663-696, (1985).

[66] M. van Hoeij, *Factorization of Differential Operators with Rational Functions Coefficients*, J. Symbolic Computation, **24**, 537-561 (1997).

[67] M. van Hoeij, *Decomposing a 4'th order linear differential equation as a symmetric product*, Banach Center Publications, **58**, 89-96, (2002).

[68] A. C. Person, *Solving Homogeneous Linear Differential Equations of Order 4 in Terms of Equations of Smaller Order*, PhD thesis, www.lib.ncsu.edu/theses/available/etd-08062002-104315/ (2002).

[69] M. van Hoeij, *Solving Third Order Linear Differential Equations in Terms of Second Order Equations*, ISSAC'07 Proceedings, 355-360, (2007). Implementation: www.math.fsu.edu/∼hoeij/files/ReduceOrder

[70] M. van Hoeij and J.A. Weil, *Solving Second Order Linear Differential Equations with Klein's Theorem*, ISSAC'05 Proceedings, 340-347, (2005). Implementation available at www.unilim.fr/pages_perso/jacques-arthur.weil/issac05

[71] J. Kovacic, *An algorithm for solving second order linear homogeneous equations*, J. Symbolic Computation, **2**, p. 3-43 (1986).

[72] D. Krammer, *An example of an arithmetic Fuchsian group*, J. reine angew. Math. **473**, 69-85 (1996).

[73] R.S. Maier, *On reducing the Heun equation to the hypergeometric equation*, J. Differential Equations, **213**, 171-203 (2005).

[74] M. van der Put, *Galois Theory of Differential Equations, Algebraic Groups and Lie Algebras*, J. Symbolic Computation **28**, 441-472 (1999).

[75] M. van der Put, M.F. Singer, *Galois Theory of linear Differential Equations*, Grundlehren der mathematischen Wissenschaften, **328**, Springer (2003).

[76] M. van der Put and M. Singer, *Galois Theory of Difference Equations*, Lecture Notes in Mathematics, **1666**, Springer-Verlag, (1997).

[77] M.F. Singer and F. Ulmer, *Linear Differential Equations and Products of Linear Forms*, J. of Pure and Applied Algebra, **117**, 549-564 (1997).

[78] F. Ulmer, *Liouvillian solutions of third order differential equations*, J. Symb. Comp., **36**, 855-889, (2003).

[79] D. Zeilberger, *The Method of Creative Telescoping.* J. Symbolic Computation, **11**, 195-204 (1991).

[80] *The On-Line Encyclopedia of Integer Sequences*, oeis.org