Solving Linear Differential Equations in terms of Hypergeometric Functions

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Differential operator and differential equation

Let

\[ L = a_n \partial^n + a_{n-1} \partial^{n-1} + \cdots + a_1 \partial + a_0 \]

be a differential operator, with \( a_n, a_{n-1}, \ldots, a_1, a_0 \in \mathbb{C}(x) \) and \( n \) positive integer. The corresponding differential equation is

\[ a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0. \]
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We are interested in finding the closed form solution of such second order differential equations.
Closed Form Solution

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**Defined Function Set**

\{ \mathbb{C}(x), \exp, \log, \text{Airy}, \text{Bessel}, \text{Kummer}, \text{Whittaker}, \text{and } \,_{2}F_{1}-\text{Hypergeometric functions} \}
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Defined Operations Set

\{\text{field operations, algebraic extensions, compositions, differentiation and } \int dx\}
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\[
\begin{align*}
\sum_{n=0}^{\infty} & \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(a+n)\Gamma(b+n)\Gamma(c+n)} \frac{z^n}{n!} \\
& = {}_{2}F_{1}(a, b | c | x)
\end{align*}
\]
Traditional Methods of Solving Differential Operator $L$

- Direct solving by the existing techniques.
- Factor $L$ as a product of lower order differential operators, then solve $L$ by solving the lower order ones.
- Solve $L$ in terms of lower order differential operator.
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In this thesis we focus on second order linear differential equations (differential operators) which are irreducible and have no Liouvillian solutions.

**Question:** For the equations that we can’t solve by the above techniques, what should we do?
Overview of the methods

We consider to reduce the differential operator $L$, if possible, to another differential operator $\tilde{L}$ that is easier to solve (with same order, but with fewer true singularities) by using the 2-descent method or other descent methods.

1. If the above 2-descent exists, we find $\tilde{L}$.

2. If the number of true singularities of $\tilde{L}$ drops to 3, we find its $2\, {}_1F_1$-type solutions, furthermore, find the $2\, {}_1F_1$ solution of $L$ in terms of $\tilde{L}$’s.

3. If the number of true singularities of $\tilde{L}$ drops to 4, we can decide if $\tilde{L}$, furthermore $\exists \ 2\, {}_1F_1$-type solutions by building a large table that covers the differential operators with 4 true singularities.
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Transformations

When we talk about that $L$ can be solved in terms of the solutions of $\tilde{L}$, we mean that $\tilde{L}$ can be transformed to $L$. 
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There are three types of transformations that preserve order 2:

1. **change of variables:** $y(x) \rightarrow y(f(x))$, $f(x) \in \mathbb{C}(x) \setminus \mathbb{C}$.
2. **exp-product:** $y \rightarrow e^{\int r \, dx} \cdot y$, $r \in \mathbb{C}(x)$.
3. **gauge transformation:** $y \rightarrow r_0 y + r_1 y'$, $r_0, r_1 \in \mathbb{C}(x)$. 

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Example 1

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Given $L_1, L_2 \in \mathbb{C}(x)[\partial]$ with order 2:

If $L_1 \xrightarrow{2\&3} L_2$, then $L_1 \sim_p L_2$ (projectively equivalent).

If $L_1 \xrightarrow{3} L_2$, then $L_1 \sim_g L_2$ (gauge equivalent).
Example 1

\[ L = x^2(36x^2 - 1)(4x^2 - 1)(12x^2 - 1)\partial^2 + 4x(2x - 1)(1296x^5 + 576x^4 - 144x^3 - 72x^2 + x + 1)\partial + 2(5184x^6 - 864x^5 - 1656x^4 + 48x^3 + 162x^2 + 6x - 1) \]

Question: How to find the \( _2F_1 \) solution of \( L \) as follows:

\[ y_1 = r_1 \cdot _2F_1 \left( \begin{array}{c} 1/4, 1/4 \\ 3/2 \end{array} \right| \frac{144x^4 + 24x^2 + 1}{64x^2} \right) \]

\[ + r_2 \cdot _2F_1 \left( \begin{array}{c} 5/4, 5/4 \\ 5/2 \end{array} \right| \frac{144x^4 + 24x^2 + 1}{64x^2} \right) \]

(with \( r_1, r_2 \in \mathbb{C}(x) \))

\[ y_2 = \cdots \]
Informal definition for 2-descent

For a second order differential operator $L$ over $\mathbb{C}(x)$, we say that $L$ has 2-descent if $L$ can be reduced to $\tilde{L}$ with the same order defined over a subfield $k \subset \mathbb{C}(x)$ with index 2.
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**Benefits for finding 2-descent of $L$**

- Reduce the number of true singularities from $n$ to $\leq \frac{n}{2} + 2$.
- Help to find the $2F_1$-type solutions.
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Compoint, van Hoeij, van der Put reduced the problem of 2-descent to another problem, which involved in trivializing a 2-cocycle.

- No explicit algorithms are given.

- van Hoeij proposed that we first compute the symmetric product of $L$ and $\sigma(L)$, and then factor it to the product of a first order equation and third order equation and then use another method to find the equivalent second order differential equation of the third order factor.

- The method here involves to calculate the point on a conic. Algorithms were only given when the conic is defined over $\mathbb{Q}$ or the transcendental of $\mathbb{Q}$. NO algorithms are given for the general ground field.
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Main Goal

Given a second order differential operator $L$, our goal is to give an explicit algorithm to decide if $L$ has 2-descent, and if so, find this descent.
Formal definition for 2-descent

Given a second order differential operator $L$ defined over $\mathbb{C}(x)$, we say that $L$ has 2-descent if $\exists f \in \mathbb{C}(x)$ with $\text{degree}(f) = 2$, and $\exists \tilde{L} \in \mathbb{C}(f)[\partial_f]$ such that $L \sim_p \tilde{L}$. 

Note: $\partial_f = \frac{d}{df} = \frac{1}{f'} \partial$. 

Two steps to achieve the main goal

1. Finding the subfield $\mathbb{C}(f)$ with $[\mathbb{C}(x) : \mathbb{C}(f)] = 2$, i.e. finding $f \in \mathbb{C}(x)$ of degree 2.
2. Finding the projectively equivalent differential operator $\tilde{L} \in \mathbb{C}(f)[\partial_f]$. 

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Finding the subfield $\mathbb{C}(f)$

Since every extension of degree 2 is Galois, so by Lüroth's theorem, we have the following relationship:
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\[ \mathbb{C}(f) \subset \mathbb{C}(x) \text{ with } [\mathbb{C}(x) : \mathbb{C}(f)] = 2 \iff \sigma \in \text{Aut}(\mathbb{C}(x)/\mathbb{C}) \text{ with degree 2} \]
Since every extension of degree 2 is Galois, so by Lüroth’s theorem, we have the following relationship:

\[
\text{Remark} \quad \sigma \in \text{Aut}(\mathbb{C}(x)/\mathbb{C}) \text{ with degree } 2
\]

The automorphisms of \(\mathbb{C}(x)\) over \(\mathbb{C}\) are Möbius transformations:

\[
x \mapsto \frac{ax + b}{cx + d}
\]
Finding the subfield $\mathbb{C}(f)$

Requirements for $\sigma$

- $\sigma = \frac{ax+b}{cx+d}$ with $d = -a$;
- $\sigma$ should preserve the set of true singularities of $L$ and their exponent-difference mod $\mathbb{Z}$. 
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- $\sigma$ should preserve the set of true singularities of $L$ and their exponent-difference mod $\mathbb{Z}$.

For each such $\sigma$, we compute a candidate subfield $\mathbb{C}(f) \subseteq \mathbb{C}(x)$. To determine $\sigma$, basically, we need find 2 equations of variables $a, b, c$ and then verify if it satisfies the requirements mentioned above.
Example 2

Let $C = \mathbb{Q}$, and

$$L = \partial^2 + \frac{(44x^4 - 7)}{x(2x^2 - 1)(2x^2 + 1)} \partial + \frac{8(24x^6 - 14x^4 - 3x^2 + 1)}{x^2(2x^2 + 1)(2x^2 - 1)^2}$$

- The set of true singularities is
  $$S = \{\infty, 0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{-2}}, \frac{1}{\sqrt{-2}}\}$$

- and
  $$S^\text{type}_C = \{(\infty, 0), (x, 0), (x^2 + \frac{1}{2}, 0), (x^2 - \frac{1}{2}, 0)\}.$$
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Analyze example 2, we get the set of candidates for $\sigma$ is:

$$\{-x, -\frac{1}{2x}, \frac{1}{2x}\}$$

The corresponding subfields set is:

$$\{ \mathbb{C}(x^2), \mathbb{C}(x - \frac{1}{2x}), \mathbb{C}(x + \frac{1}{2x}) \}$$
The following $\sigma$ and $\mathbb{C}(f)$ represent the Möbius transformation found previously and the corresponding fixed field, respectively. Suppose $L$ descends to $\tilde{L} \in \mathbb{C}(f)[\partial_f]$, we have

$$L \sim_p \tilde{L} = \sigma(\tilde{L}) \sim_p \sigma(L),$$

and so $L \sim_p \sigma(L)$.
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which means we can find the projective equivalence:

$$y \rightarrow e^{\int r \, dx} \cdot (r_0 y + r_1 y')$$

from the solution space of $L$ to the solution space of $\sigma(L)$.
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which means we can find the projective equivalence:

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from the solution space of $L$ to the solution space of $\sigma(L)$.

Question: How to compute $\tilde{L}$ from it?
Case A is when $L \sim_g \sigma(L)$, in other words, there exists $G = r_0 + r_1 \partial \in \mathbb{C}(x)[\partial]$ with $G(V(L)) = V(\sigma(L))$. Then $\exists \tilde{L} \in \mathbb{C}(f)[\partial_f]$ with $\tilde{L} \sim_g L$. 

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Question: Given $G$, how to find $\tilde{L}$?

\[ V(L) \xrightarrow{A} V(\tilde{L}) = V(\sigma(\tilde{L})) \]
Question arising in the above diagram

Question: When does the above diagram commute?
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**Theorem**

Let $L$ and $\sigma$ be as before, and $G : V(L) \rightarrow V(\sigma(L))$ be a gauge transformation. Suppose $\tilde{L}_1, \tilde{L}_2 \in \mathbb{C}(f)[\partial_f]$ and $A_i : V(L) \rightarrow V(\tilde{L}_i)$ are gauge transformations. Then:

1. For each $i = 1, 2$, there is exactly one $\lambda_i \in \mathbb{C}^*$ such that the following diagram commutes.

2. If $\tilde{L}_1 \sim_g \tilde{L}_2$ over $\mathbb{C}(f)$, then $\lambda_1 = \lambda_2$; Otherwise, $\lambda_1 = -\lambda_2$.

3. In particular, $\{\lambda_1, -\lambda_1\}$ depends only on $(L, \sigma, G)$. 
Finding the projectively equivalent operator $\tilde{L}$

Diagram

$$V(L) \xrightarrow{\lambda_i G} V(\sigma(L))$$

$$V(\tilde{L}_i) \xleftarrow{\sigma(A_i)}$$
Finding the projectively equivalent operator $\tilde{L}$

Finding $\tilde{L}$ in Case A

$V(L) \xrightarrow{\lambda G} V(\sigma(L))$

$\xrightarrow{A}$

$\xrightarrow{\sigma(A)}$

$V(\tilde{L})$

$A^{-\sigma(A)} \lambda G$ becomes a map from $V(L)$ to $V(\tilde{L})$, and has a nonzero kernel. This kernel corresponds to a right hand factor of $L$, since $L$ is irreducible, the kernel is $V(L)$ itself.

$A^{-\sigma(A)} \lambda G$ right divided by $L$, and this gives us 4 equations for coefficients of $A$. 

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Finding $\tilde{L}$ in Case A

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Finding $\tilde{L}$ in Case A

$V(L)$ \xrightarrow{\lambda G} V(\sigma(L)) \xrightarrow{\sigma(A)} V(\tilde{L})$

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$A - \sigma(A)\lambda G$ right divided by $L$, and this gives us 4 equations for coefficients of $A$. 
Example 3

\[ L = \partial^2 + \frac{8(8x+1)}{(4x+1)(4x-1)} \partial + \frac{4(8x+1)}{x(4x-1)(4x+1)}. \]

One of the candidates we found for \( \sigma \) is \(-x\) and
\[ G = \frac{x(4x-1)}{4x+1} \partial + \frac{12x+1}{2(4x+1)}. \]

We implement the algorithm as follows:

- Write \( A = (a_{10} + a_{11}x)\partial + (a_{00} + a_{01}x) \), with \( a_{00}, a_{01}, a_{10}, a_{11} \) unknown and over \( \mathbb{C}(f) \).
- Get \( \sigma(A) = -(a_{10} - a_{11}x)\partial + a_{00} - a_{01}x \). Set the remainder of \( A - \sigma(A)\lambda G \) right divided by \( L \) to be 0. We get a set of the coefficients as:

\[
\begin{align*}
2a_{01} - 16\lambda a_{00} + \lambda a_{01} - 64\lambda a_{10} + 32fa_{10} + 48f\lambda a_{01} + 16a_{00}, \\
16fa_{01} + 2a_{00} + 32fa_{00} + 64f\lambda a_{11} - \lambda a_{00} - 48f\lambda a_{00} + 16f\lambda a_{01}, \\
16\lambda a_{10} + 2\lambda a_{00} + 32fa_{11} + 48f\lambda a_{11} - 32f\lambda a_{00} + 16a_{10} + \\
\lambda a_{11} + 2a_{11}, -16f\lambda a_{11} + 2a_{10} + 32f^2\lambda a_{01} + 16fa_{11} - 48f\lambda a_{10} - \\
\lambda a_{10} + 32fa_{10} - 2f\lambda a_{01}.
\end{align*}
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  \[ \{ 2a_{01} - 16\lambda a_{00} + \lambda a_{01} - 64\lambda a_{10} + 32fa_{10} + 48f \lambda a_{01} + 16a_{00}, 16fa_{01} + 2a_{00} + 32fa_{00} + 64f \lambda a_{11} - \lambda a_{00} - 48f \lambda a_{00} + 16f \lambda a_{01}, 16\lambda a_{10} + 2\lambda a_{00} + 32fa_{11} + 48f \lambda a_{11} - 32f \lambda a_{00} + 16a_{10} + \lambda a_{11} + 2a_{11}, -16f \lambda a_{11} + 2a_{10} + 32f^2 \lambda a_{01} + 16fa_{11} - 48f \lambda a_{10} - \lambda a_{10} + 32fa_{10} - 2f \lambda a_{01} \}. \]
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\[
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\]
Example 3, continued...

- Equate the determinant of the corresponding matrix $M$ to 0 gives a degree 4 equation for $\lambda$. Solve for $\lambda$.

- Plug in one value for $\lambda$ in $M$, then solve $M$ to find values for $a_{00}, a_{01}, a_{10}, a_{11}$ in $A$. We take $\lambda = 2$ and get

$$A = \left(\frac{4}{3}x^2 - \frac{1}{12}\right)\partial + \frac{4x}{3} + 1$$

- Implement the Maple Command LCLM of $A$ and $L$, and then the Command rightdivision of the result gotten just now by $A$, we get the 2-descent $\tilde{L}$:

$$\tilde{L} = (16x_1 - 1)x_1\partial^2 + (32x_1 - 1)\partial + 4$$
Example 3, continued...

- Equate the determinant of the corresponding matrix $M$ to 0 gives a degree 4 equation for $\lambda$. Solve for $\lambda$.

  $\det(M) = 65536(\lambda - 2)^2 \cdot (\lambda + 2)^2 \cdot (f - 1/16)^4$ and $\lambda = \pm 2$.

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Case B is when \( L \sim_p \sigma(L) \), in other words, there exists \( G = e^{\int r} \cdot (r_0 + r_1 \partial) \) such that \( G(V(L)) = V(\sigma(L)) \).
Case B

**Case B** is when $L \sim_p \sigma(L)$, in other words, there exists $G = e^{ \int r \cdot (r_0 + r_1 \partial) }$ such that $G(V(L)) = V(\sigma(L))$.

**Difficulty**

We have an exponential part in $G$ comparing with **Case A**. The algorithm mentioned above fails.
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We have an exponential part in $G$ comparing with **Case A**. The algorithm mentioned above fails.

**Solution**

After multiplying solution of $L$ by a suitable $e^{\int s}$, we can reduce this case to Case A.
Sketch of the Main Algorithm

Main Algorithm:

Input: A second order differential operator $L$; 
Output: Another second order differential operator $\tilde{L}$.

1. Compute the set of true singularities of $L$, and their exponent-difference mod $\mathbb{Z}$.
2. Compute the candidates set for $\sigma$.
3. For each $\sigma$, check if $L \sim_p \sigma(L)$, and if so, to find $G : V(L) \to V(\sigma(L))$.
4. If we find $\sigma$ with $L \sim_g \sigma(L)$, then call algorithm Case A and stop; otherwise, if $L \sim_p \sigma(L)$ reduce Case B to Case A.
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4. If we find $\sigma$ with $L \sim_g \sigma(L)$, then call algorithm Case A and stop; otherwise, if $L \sim_p \sigma(L)$ reduce Case B to Case A.
To decide $\tilde{L}$, we first compute $\lambda$ and then a set of linear equations to determine $A = (a_{10} + a_{11}x)\partial + (a_{00} + a_{01}x)$. 

**Advantage**

This algorithm does give us one $\tilde{L}$ which is equivalent to our input $L$.

**Disadvantage**

When we compute $A$, we select one $(a_{00}, a_{01}, a_{10}, a_{11})$ from a vector space of dimension 2, that means our output $\tilde{L}$ is just one member of a 2-dimensional set of possible outcomes. We can't expect $\tilde{L}$ to have the optimal size.
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What is improved in the new Algorithm

The improved algorithm will avoid computing a set of possible $\tilde{L}$s and apt to give a smaller output.
Main Idea for the Improved Case A algorithm

We consider the following algorithm, here we denote
\[ L_4 := \text{LCLM}(L, \sigma(L)) \in C(f)[\partial_f] \] then
\[ V(L_4) = V(L) + V(\sigma(L)). \]
The order of \( L_4 \) is 4 except if \( V(L) = V(\sigma(L)) \).
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\[ V(L) \xrightarrow{1+G} \] 
\[ \sigma(1+G) \xrightarrow{V(L_4)} V(\sigma(L)) \]
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\[
\begin{align*}
V(L) \quad \xrightarrow{1+G} \quad V(L_4) \\
\downarrow \quad \quad \quad \quad \downarrow \\
V(\sigma(L)) \quad \quad \quad \xrightarrow{\sigma(1+G)}
\end{align*}
\]
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Question: Is this a commutative diagram?
Support Theorem

Given a second order irreducible differential operator $L$ and second order automorphism $\sigma$ as in Section 3.4, and a gauge transformation $G : V(L) \rightarrow V(\sigma(L))$, then there exist a constant $\lambda$ such that the following diagram commutes.

$$V(L) \xrightarrow{\lambda G} V(\sigma(L))$$

$$V(L^4) \xrightarrow{\lambda \sigma(G) + 1} 1 + \lambda G$$

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**Support Theorem**

**Lemma**

Given a second order irreducible differential operator $L$ and second order automorphism $\sigma$ as in Section 3.4, and a gauge transformation $G : V(L) \rightarrow V(\sigma(L))$, then there exist a constant $\lambda$ such that the following diagram commutes.
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Lemma

Given a second order irreducible differential operator $L$ and second order automorphism $\sigma$ as in Section 3.4, and a gauge transformation $G : V(L) \rightarrow V(\sigma(L))$, then there exist a constant $\lambda$ such that the following diagram commutes.

$$
\begin{align*}
V(L) & \xrightarrow{\lambda G} V(\sigma(L)) \\
& \xleftarrow{1 + \lambda G} V(L_4) \\
& \xrightarrow{\lambda \sigma(G) + 1}
\end{align*}
$$
Finding $\tilde{L}$

Questions: with the above diagram, do we have the $\tilde{L}$ already? Is $L_4$ the descent we want? If not, how do we find $\tilde{L}$?
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**Theorem**

Given a second order differential operator $L$. $\sigma$, $G$, $\lambda$ are as stated previously in the diagram. Then there exists a second order differential operator $\tilde{L}$ such that $\tilde{L}$ is invariant under $\sigma$ and $(1 + \lambda G)V(L) = V(\tilde{L})$. 

Computing $\tilde{L}$

1. Compute $M := LCLM(L_1 + \lambda G)$
2. Compute the $\tilde{L}$ such that $M = \tilde{L}(1 + \lambda G)$
3. Verify $V(\tilde{L}) \subseteq V(L_4)$ and $V(\sigma(\tilde{L})) = V(\tilde{L})$
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**Computing $\tilde{L}$**

1. Compute $M := \text{LCLM}(L, 1 + \lambda G)$.
2. Compute the $\tilde{L}$ such that $M = \tilde{L}(1 + \lambda G)$.
3. Verify $V(\tilde{L}) \subseteq V(L_4)$ and $V(\sigma(\tilde{L})) = V(\tilde{L})$.
Application for Fourth Order Differential Equation

\[ L := \partial^4 + \frac{(7x^4 - 68x^3 - 114x^2 + 52x - 5)}{(x + 1)(x^2 - 10x + 1)(x - 1)x} \partial^3 + \frac{2(5x^5 - 55x^4 - 169x^3 + 149x^2 - 28x + 2)}{(x + 1)x^2(x^2 - 10x + 1)(x - 1)^2} \partial^2 + \frac{2(x^4 - 13x^3 - 129x^2 + 49x - 4)}{(x + 1)x^2(x^2 - 10x + 1)(x - 1)^2} \partial^- + \frac{3(x + 1)^2}{(x - 1)^2x^3(x^2 - 10x + 1)} \partial^{'} \]

\[ L \] has 4 regular true singularities:
Result after 2-descent Algorithm, Case A

\[ \tilde{\mathcal{L}}_1 := 16x_1^4(x_1 + 3)(5x_1^2 + 10x_1 + 1)(9x_1^8 + 1008x_1^7 - 31820x_1^6 + 264480x_1^5 \\
- 14194x_1^4 + 162992x_1^3 - 8156x_1^2 + 18368x_1 + 529)(x_1 - 1)^4 \partial^4 \\
+ 32x_1^3(-7935 - 358000x_1 - 3502550x_1^2 - 24264785x_1^4 - 1520720x_1^3 \\
- 12737440x_1^5 - 13562976x_1^7 - 20800372x_1^6 - 905046x_1^{10} + 20706063x_1^8 \\
+ 28080x_1^{11} + 6593808x_1^9 + 225x_1^{12})(x_1 - 1)^3 \partial^3 \\
+ 8x_1^2(2250x_1^{13} + 312135x_1^{12} - 12439492x_1^{11} + 134614866x_1^{10} \\
- 42449802x_1^9 - 470021643x_1^8 + 267358792x_1^7 - 102361428x_1^6 + 163767350x_1^5 \\
+ 221768417x_1^4 - 11134724x_1^3 + 48114210x_1^2 + 3717898x_1 + 77763)(x_1 - 1)^2 \partial^2 \\
+ 8x_1(x_1 - 1)(1350x_1^{14} + 230355x_1^{13} - 10741153x_1^{12} + 169118578x_1^{11} \\
- 503407892x_1^{10} + 340703465x_1^9 + 768939585x_1^8 - 411403540x_1^7 \\
+ 839007558x_1^6 - 333028107x_1^5 - 52500447x_1^4 + 44391810x_1^3 - 43359960x_1^2 \\
- 2602385x_1 - 42849) \partial + \cdots \]
Result after Improved 2-descent Algorithm, Case A

\[ \tilde{L}_2 := \partial^4 + \frac{77x_1^6 - 1709x_1^5 - 11250x_1^4 - 11530x_1^3 + 10377x_1^2 - 2457x_1 + 108}{(x_1 - 1)x_1(11x_1^5 - 215x_1^4 - 1250x_1^3 - 1278x_1^2 + 711x_1 - 27)} \partial^3 + \]
\[ \frac{220x_1^7 - 6063x_1^6 - 46066x_1^5 - 40985x_1^4 + 71024x_1^3 - 30225x_1^2 + 3078x_1 - 135}{2(x_1^2 - 2x_1 + 1)x_1^2(11x_1^5 - 215x_1^4 - 1250x_1^3 - 1278x_1^2 + 711x_1 - 27)} \partial^2 + \]
\[ \frac{22x_1^6 - 931x_1^5 - 10011x_1^4 - 12590x_1^3 + 15680x_1^2 - 3039x_1 + 117}{16(x_1^2 - 2x_1 + 1)x_1^4(11x_1^4 - 248x_1^3 - 506x_1^2 + 240x_1 - 9)} \partial - \]
\[ 3(121x_1^5 + 175x_1^4 - 166x_1^3 + 1118x_1^2 - 227x_1 + 3) \]

Where \( x_1 \) represents \( x^2 \). \( \tilde{L}_2 \) has length 635.
After implementing 2-decent, we may end up with \( \tilde{L} \) with 3 true singularities. If so, we can solve such \( \tilde{L} \) in terms of hypergeometric functions, further more \( L \).

To find the \( {}_2F_1 \) Solutions, we need connect with the hypergeometric equations, which have the following properties: 

(a) Three true regular singularities, located at 0, 1, \( \infty \).
(b) No apparent singularities.
Things we should consider

After implementing 2-decent, we may end up with $\tilde{L}$ with 3 true singularities. If so, we can solve such $\tilde{L}$ in terms of hypergeometric functions, further more $L$.

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What we have for $\tilde{L}$?

(a) Three true regular singularities, located say at $p_1, p_2, p_3 \in \mathbb{P}^1$.

(b) Any number of apparent singularities.
What we have for $\tilde{L}$?

(a) Three true regular singularities, located say at $p_1, p_2, p_3 \in \mathbb{P}^1$.

(b) Any number of apparent singularities.

To solve $\tilde{L}$ in terms of hypergeometric functions, we need to apply two types of transformations:

(a) A Möbius transformation (a change of variables) to move $p_1, p_2, p_3$ to $0, 1, \infty$.

(b) A projective equivalence $\sim_p$ to eliminate all apparent singularities.
Classification of Gauss Hypergeometric Equations

Theorem

Let $L_1, L_2$ be two Gauss hypergeometric differential operators. Assume the exponent difference set of $L_1$ at $0, 1, \infty$ is $\{e_0, e_1, e_\infty\}$, and the exponent difference set of $L_2$ at $0, 1, \infty$ is $\{d_0, d_1, d_\infty\}$. If $1 - d_i \in \mathbb{Z}$ for all $i \in \{0, 1, \infty\}$ and $2 \sum_{i \in \{0, 1, \infty\}} (e_i - d_i)$ is an even integer, then $L_1 \sim p L_2$.

Corollary

Let $L_1, L_2$ be two Gauss hypergeometric differential operators. Assume the exponent difference set of $L_1$ at $0, 1, \infty$ is $\{e_0, e_1, e_\infty\}$, and the exponent difference set of $L_2$ at $0, 1, \infty$ is $\{d_0, d_1, d_\infty\}$. If $2 \in \mathbb{Z}$ appears in $\{e_0, e_1, e_\infty\}$ and $\{d_0, d_1, d_\infty\}$, then $L_1$ is projectively equivalent to $L_2$ if $e_i - d_i \in \mathbb{Z}$ for all $i \in \{0, 1, \infty\}$. 
Theorem

Let $L_1, L_2$ be two Gauss hypergeometric differential operators. Assume the exponent difference set of $L_1$ at 0, 1, $\infty$ is \{e_0, e_1, e_\infty\}, and the exponent difference set of $L_2$ at 0, 1, $\infty$ is \{d_0, d_1, d_\infty\}. If

1. \(e_i - d_i \in \mathbb{Z}\) for all \(i \in \{0, 1, \infty\}\)

and

2. \(\sum_{i \in \{0,1,\infty\}} (e_i - d_i)\) is an even integer,

Then $L_1 \sim_p L_2$. 

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1. $e_i - d_i \in \mathbb{Z}$ for all $i \in \{0, 1, \infty\}$

and

2. $\sum_{i \in \{0, 1, \infty\}} (e_i - d_i)$ is an even integer,

Then $L_1 \sim_p L_2$.

Corollary

Let $L_1, L_2$ be two Gauss hypergeometric differential operators. Assume the exponent difference set of $L_1$ at $0, 1, \infty$ is $\{e_0, e_1, e_\infty\}$, and the exponent difference set of $L_2$ at $0, 1, \infty$ is $\{d_0, d_1, d_\infty\}$. If $\frac{1}{2} + \mathbb{Z}$ appears in $\{e_0, e_1, e_\infty\}$ and $\{d_0, d_1, d_\infty\}$, then $L_1$ is projectively equivalent to $L_2$ if $e_i - d_i \in \mathbb{Z}$ for all $i \in \{0, 1, \infty\}$.
Possible Hypergeometric Equations corresponding to $\tilde{L}$

Lemma

Suppose $L$ is projectively equivalent to a hypergeometric equation. Suppose that the exponent-differences of $L$ at $0, 1, \infty$ are $d_0, d_1, d_\infty$. Let $L_1$ be a hypergeometric equation with exponent-differences: $d_0, d_1, d_\infty$ and $L_2$ be a hypergeometric equation with exponent-differences: $d_0 + 1, d_1, d_\infty$. Then $L \sim_p L_1$ or $L \sim_p L_2$ (both are true if $\{d_0, d_1, d_\infty\} \cap \{\frac{1}{2} + \mathbb{Z}\} \neq \emptyset$).
Possible Hypergeometric Equations corresponding to $\tilde{L}$

**Lemma**

Suppose $L$ is projectively equivalent to a hypergeometric equation. Suppose that the exponent-differences of $L$ at $0, 1, \infty$ are $d_0, d_1, d_\infty$. Let $L_1$ be a hypergeometric equation with exponent-differences: $d_0, d_1, d_\infty$ and $L_2$ be a hypergeometric equation with exponent-differences: $d_0 + 1, d_1, d_\infty$. Then $L \sim_p L_1$ or $L \sim_p L_2$ (both are true if $\{d_0, d_1, d_\infty\} \cap \{\frac{1}{2} + \mathbb{Z}\} \neq \emptyset$).

With these exponent-differences $d_0, d_1, d_\infty$ at $0, 1, \infty$, we construct the gauss hypergeometric equations by the following formula:

$$x(x-1)\partial^2 - (-2x + xd_0 + xd_1 + 1 - d_0)\partial + \frac{(d_0 - 1 + d_1 + d_\infty)(d_0 - 1 + d_1 - d_\infty)}{4}$$
Having these theorems, we have evidences to find the $2F_1$ solution of our $\tilde{L}$.

1. Compute the exponent-difference at the three singularities of $\tilde{L}$ module $\mathbb{Z}$. Denote them as $e_1, e_2, e_3$.

2. Find the two Gauss hypergeometric equations $L_1, L_2$ by the formula and theorem.

3. Find the Möbius transformation $m(x)$ between $p_1, p_2, p_3$ and $0, 1, \infty$.

4. Call `equiv` to check if $L_1$ or $L_2$ (after change of variable) is projectively equivalent to $\tilde{L}$, if so, go to next step. Denote the equivalence as $G$.

5. Find the Gauss hypergeometric solutions of
   
   $Sol := C_1 y_1(m(x)) + C_2 y_2(m(x))$ if $e_1 \neq 0$, otherwise, compute
   
   $Sol := C_1 y_1(m(x)) + C_2 y_2'(m(x))$.

6. Compute the $2F_1$-type solution of $\tilde{L}$ by computing $G(Sol)$.
Final solving by 2-descent

**Input:** A second order irreducible differential operator $L \in \mathbb{C}(x)[\partial]$ and the field $\mathbb{C}$.

**Output:** $2F_1$-type solution, if it exists.

1. Call **Algorithm 2-descent** in Chapter 3 to Compute the 2-descent of $L$, $\tilde{L}$, if it exists.

2. Compute the true singularities of $\tilde{L}$.

3. If $\tilde{L}$ has 3 true regular singularities, then call **Algorithm finding $2F_1$-type solution with 3 singularities** and find the solution $sol$; Otherwise, stop and return NULL.

4. Apply the Change of variable $x \mapsto f$ to $\tilde{L}$, $Sol$, we get $\tilde{L}'$ and its $2F_1$ solution $Sol'$.

5. Call **equiv** to Compute the equivalence $G$ between $\tilde{L}'$ and $L$.

6. Compute the $2F_1$-type solution of $L$ by computing $G(Sol')$. 
Example 5

Let

$$L = \partial^2 + \frac{28x - 5}{x(4x - 1)} \partial + \frac{144x^2 + 20x - 3}{x^2(4x - 1)(4x + 1)}$$

Step 1: Compute the 2-descent of $L$ from Section 3.7, we have

$$\tilde{L} := (16x - 1)x\partial^2 + (32x - 2)\partial + 4$$

Step 2: Compute the true singularities of $\tilde{L}$, we found it has 3 true regular singularities: $0, \frac{1}{16}, \infty$.

Step 3: Call Algorithm finding $\, _2F_1$-type solution with 3 singularities, we found the $\, _2F_1$ solution of $\tilde{L}$ as

$$\text{Sol} := C_1(64x-4)\, _2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; 16x\right) - C_2(64x-4)\, _2F_1\left(\frac{3}{2}, \frac{3}{2}; 2; 1-16x\right)$$
Example 5, continued...

**Step 4:** From Section 3.7, we know that \( f = x^2 \), so the change of variable would be \( x \mapsto x^2 \). Apply transformation to \( \tilde{L} \) and \( \text{Sol} \), we have

\[
\tilde{L}' := x(4x + 1)(4x - 1)\partial^2 + (12x - 3)(4x + 1)\partial + 16x
\]

\[
\text{Sol}' := C_1(64x^2 - 4)_{2F1}(\frac{3}{2}, \frac{3}{2}; 2; 16x^2) - C_2(64x^2 - 4)_{2F1}(\frac{3}{2}, \frac{3}{2}; 2; 1 - 16x^2)
\]

**Step 5:** Compute the equivalence between \( \tilde{L}' \) and \( L \), we have

\[
G := \frac{1}{x(4x - 1)}
\]

**Step 6:** Compute \( G(\text{Sol}') \), we have the final solution as

\[
C_1 \frac{16x + 1}{x}_{2F1}(\frac{3}{2}, \frac{3}{2}; 2; 16x^2) - C_2 \frac{16x + 1}{x}_{2F1}(\frac{3}{2}, \frac{3}{2}; 2; 1 - 16x^2)
\]
We focus on finding the hypergeometric solutions of second order linear equations. Contributions of this thesis are:

1. Developed 2-descent algorithms to reduce our differential equation to one with fewer true singularities.
2. Improved the 2-descent algorithm to produce shorter output, which is helpful for finding the $2F_1$ solutions.
3. Finding the $2F_1$ solutions.

Work may be done in future:

1. Extend the 2-descent algorithm to bigger descent, for example: $C_2 \times C_2$, $D_n$, $A_4$, $S_4$, or $A_5$.
2. Extend the 2-descent to 3-descent, for which the index of the descent subfield is 3.
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