Finite Singularities of Linear Difference Operators Extended Abstract.

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1 Differential and difference operators

Differential equations and recurrence equations have a number of resemblances. We will show how finite singularities can be used to mimic methods for differential equations in the case of recurrence equations.

A linear homogeneous differential equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

corresponds to a *differential operator*

$$f = a_n \partial^n + a_{n-1} \partial^{n-1} + \cdots + a_0 \partial^0$$

acting on y. The coefficients a_i that are considered here are elements of the differential field $\mathbb{C}(x)$ and ∂ is a differentiation d/dx. The differential operator f is an element of the non-commutative ring $\mathbb{C}(x)[\partial]$, which is an *Ore ring*, cf. [12]. Multiplication in this ring corresponds to composition of operators. A factorization $f = LR$ where $L, R \in \mathbb{C}(x)[\partial]$ is useful for computing solutions of f because solutions of the right-hand factor R are solutions of f as well. Right-hand factors of order 1 are particularly interesting because these are easy to solve. Such factors $\partial - r$ where $r \in \mathbb{C}(x)$ corresponds to an *exponential solution* $\exp(\int r)$. Multiplying f on the left by $1/a_n$ does not change the solution space, so without loss of generality we may assume that $a_n = 1$, i.e. f is *monic*.

Similarly one can also study *difference operators*

$$f = a_n \tau^n + a_{n-1} \tau^{n-1} + \cdots + a_0 \tau^0$$

where τ maps x to $x+1$. Such a difference operator corresponds to a *difference equation* or *recurrence equation*

$$f(u) = a_n u(x+n) + a_{n-1} u(x+n-1) + \cdots + a_0 u(x) = 0.$$

Often one is interested in solutions that are sequences: $u(0), u(1), u(2), \ldots$, so then u is a function from $\mathbb{N}$ to $\mathbb{C}$. It will be useful to consider other type of solutions as well. If u is a solution of a difference equation of order 1, so $u(x+1) = r(x)u(x)$ for some rational function $r \in \mathbb{C}(x)$, then u is called *hypergeometric*. Just like exponential solutions in the differential case, hypergeometric solutions correspond to first order right-hand factors $\tau - r \in \mathbb{C}(x)[\tau]$ of f.

2 Finding exponential solutions

Consider the function $\exp(\int r)$ where $r \in \mathbb{C}(x)$. How to find solutions of this type of a monic operator $f = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_0\partial^0$? In [9] two methods are given, the classical method that dates from the previous century, and a new method. The classical method can be explained as follows. The function $\exp(\int r)$ is meromorphic at all but a finite number of points $p \in P^1 = \mathbb{C} \bigcup \{\infty\}$. The "non-meromorphicness" of $\exp(\int r)$ at a point $p \in P^1$ can be classified by the so-called *exponential part* which (in this case) is an element of $\mathbb{C}[x^{-1}]/\mathbb{Z}$. For example the function $x^{1/3} = \exp(\int \frac{1/3}{x})$ has exponential part $1/3 + \mathbb{Z}$ at the point 0. If $e \in \mathbb{C}[x^{-1}]$ then $\exp(\int \frac{e}{x})$ has exponential part $e + \mathbb{Z}$. The collection of the exponential parts of $\exp(\int r)$ at all points $p \in P^1$ corresponds 1–1 with the *type* of $\exp(\int r)$. The type describes the "non-meromorphicness" of $\exp(\int r)$. Two exponential functions $\exp(\int r_1)$ and $\exp(\int r_2)$ have the same type if and only if their quotient is a rational function, so if and only if $r_1 - r_2 = y'/y$ for some $y \in \mathbb{C}(x)$.

At every singularity $p \in P^1$ of the operator p one can compute the "non-meromorphicness" of all local solutions, one can compute all exponential parts at p. The number of different exponential parts at p is at most the order of the operator f. There are only finitely many singularities, and at regular points only the trivial exponential part occurs. This leaves only a finite number (but possibly exponentially large number) of cases for the type of an exponential solution $\exp(\int r)$. When the type of an exponential solution s is known, then s is known up to a rational factor, so $s = y \cdot \exp(\int r)$ for some known $r \in \mathbb{C}(x)$ and unknown $y \in \mathbb{C}(x)$. The $\mathbb{C}(x)$-automorphism of $\mathbb{C}(x)[\partial]$ given by $\partial \mapsto \partial + r$ transforms f into an operator that has as solution space the solutions of f divided by $\exp(\int r)$. This way, finding s is reduced to finding rational solutions $y \in \mathbb{C}(x)$. So computing exponential solutions can be done by making a finite list of possible types, and to compute rational solutions for each case.

3 Finding hypergeometric solutions

Similar to the differential case, computing hypergeometric solutions is equivalent to computing first order factors $\tau - r \in \mathbb{C}(x)[\tau]$ of a difference operator f. Two hypergeometric solutions u_1 and u_2 have the same type if their quotient is a rational function. These u_1 and u_2 are solutions of $\tau - r_1$ and $\tau - r_2$ where $r_i = \tau(u_i)/u_i$. The two operators $\tau - r_1$ and $\tau - r_2$ have the same type (meaning that their solutions have the same type) if $r_1/r_2 = \tau(y)/y$ for some $y \in \mathbb{C}(x)^* = \mathbb{C}(x) \setminus \{0\}$. One can compute local information about solutions at the point ∞, cf. [4, 6]. This is similar to the differential case. In the differential case all singularities are needed for determining the possible types. Likewise, in the difference case, the singularity at infinity is not sufficient. Finite singularities are needed as well.

In the differential case, to compute locally at a singularity p is done algebraically by replacing the differential field $\mathbb{C}(x)$ by $\mathbb{C}((x-p))$ if $p \in \mathbb{C}$ or $\mathbb{C}((1/x))$ if $p = \infty$. In the difference case the $\mathbb{C}(x)$-automorphism τ which maps x to $x+1$ can be extended to $\mathbb{C}((1/x))$ but not to $\mathbb{C}((x-p))$. As a consequence, although computing at the point infinity is very similar in the difference and in the differential case, for finite p a construction similar to the differential case does not work in the difference case. This is why finite singularities of difference equations have not been studied until [10]; a completely different approach is needed for this case. The key to find the definition of finite singularities is to study what is needed to be able to mimic methods (in particular: exponential solutions) from the differential case to the case of difference operators. In the differential case, the exponential parts determine the type of an exponential solution. Once the types are known, computing exponential solutions becomes easy. What kind of local data is needed to determine the type of a hypergeometric solution of a difference operator, or equivalently, the type of a first order right-hand factor?

Consider the following example: $u(n) = n! = \Gamma(n+1)$, then u is a solution of $\tau - (x+1)$. At negative integers, u has a pole of order 1 (the valuation is -1) and at positive integers the valuation is 0 (i.e. u has no pole and no root at positive integers). So, going through the integers from the left to the right, the valuation increases by 1. For $p \in \mathbb{C}/\mathbb{Z}$ the *valuation growth* of u at p is 1 at $p = 0 + \mathbb{Z} \in \mathbb{C}/\mathbb{Z}$, and it is 0 otherwise (because if $n \notin \mathbb{Z}$ then the valuation of u at n is 0). At the point infinity one can compute local data that resembles the exponential parts in the differential case. It turns out that the valuation growths at each element of $\mathbb{C}/\mathbb{Z}$ combined with local data at infinity determines the type of u. So by determining the possible valuation growths and computing some local data at infinity it is possible to give a finite list of possible types of hypergeometric solutions. Similar to differential equations, for each possible type in this list one needs to compute rational solutions of a certain transformation of the operator f. In [10] the behavior of L at $p \in \mathbb{C}/\mathbb{Z}$ is studied by considering a deformation of L. This way a finite set $\overline{g}_p(L)$ can be defined and computed, and it is shown that for all hypergeometric solutions of L the valuation growth at p is an element of $\overline{g}_p(L)$. (the converse is not true, not every element of this set needs to occur as a valuation growth of a hypergeometric solution). This way a finite list of possible types can be determined.

4 Finding rational solutions

Consider for example the following difference operator

$$\tau^2 - 2\,\frac{(x+101)\,(x-99)\,\tau}{(x+102)\,(x-98)} + \frac{(x-100)\,(x+100)}{(x+102)\,(x-98)}.$$

A basis for the rational solutions is 1/(x-100), 1/(x+100). To compute rational solutions one first needs to compute a bound for the denominator, that is a polynomial $D \in \mathbb{C}[x]$ such that the denominator of every possible solution divides D. Or alternatively one can compute a rational function Q such that every rational solution is a polynomial times Q.

In the current algorithm (cf. [1]) in Maple for rational solutions the polynomial D will have degree 201, $D = (x-100)(x-99)(x-98)\cdots(x+100)$. Using the approach from [10] one can find a method [11] to compute the denominator $D = (x-100)(x+100)$. With this smaller denominator, the numerator that needs to be computed is also much smaller and hence the computation will be faster. The computation time depends strongly on the bound for the denominator, so having a better bound gives a significant improvement. I am currently working on the implementation.

5 Other applications of finite singularities

The method given in [9] is for complicated operators more efficient than the classical method sketched in section 2. It can in principle also be applied for the difference case, however, it does not give a complete algorithm for this case. The reason is the following: The method needs to compute at one singularity, but not every singularity is suitable. In the differential case, if a singularity is not suitable a different singularity can be chosen, but in the difference case only the singularity at infinity behaves similar to the differential case. If that singularity is not suitable then the method does not work. This problem could possibly be fixed using the new finite singularities, however, there still remain a number of difficulties in order to make the method complete.

In [16] a different factorization method is given. It can factor differential operators for which the solution space is a direct sum of solution spaces of smaller operators. Computationally this method comes down to computing rational solutions of a large system of linear differential equations. One can make use of the structure of this system to compute the solutions in an efficient way [8]. In this paper the bounds on the denominators of the solutions of this large system are constructed from information of the original operator (which is much smaller and so the computation is faster). This method needs to use the *generalized exponents* at each singularity. It would be interesting to see if the finite singularities of difference equations allow a similar method. Another area of interest is the computation of the Galois group of a recurrence equation.

6 Examples

6.1 An example in Maple

Consider the following difference operator $L \in \mathbb{C}(x)[\tau]$

$$L := (3\,x+8)\,(3\,x+10)\,(x+3)\,(x+2)\,(x+1)\,\tau^3 - 3\,(3\,x+5)\,(3\,x+7)\,(x+2)\,(x+1)\,\tau^2 \\ + 3\,(3\,x+2)\,(3\,x+4)\,(x+1)\,\tau - (3\,x-1)\,(3\,x+1)$$

L corresponds to the following recurrence equation:

```
> rec_equation:=add(coeff(L,tau,i)*u(x+i),i=0..3)=0;
```

$$\mathit{rec\_equation} := -(3\,x-1)\,(3\,x+1)\,\mathrm{u}(x) + 3\,(3\,x+2)\,(3\,x+4)\,(x+1)\,\mathrm{u}(x+1) \\ - 3\,(3\,x+5)\,(3\,x+7)\,(x+2)\,(x+1)\,\mathrm{u}(x+2) \\ + (3\,x+8)\,(3\,x+10)\,(x+3)\,(x+2)\,(x+1)\,\mathrm{u}(x+3) = 0$$

Solve this equation with Maple 5.4

```
> with(LREtools): simplify(hypergeomsols(rec_equation,u(x),{}));
```

$$\frac{1}{2}\,\frac{-2\,\mathrm{u}(0) - 16\,x^2\,\mathrm{u}(1) - x^2\,\mathrm{u}(0) + 70\,x^2\,\mathrm{u}(2) - 70\,x\,\mathrm{u}(2) + 3\,x\,\mathrm{u}(0) + 32\,x\,\mathrm{u}(1)}{(3\,x-1)\,(3\,x+1)\,\Gamma(x+1)}$$

Simplify the result further as follows:

```
> u(x)=collect(numer("),{u(0),u(1),u(2)},factor)/denom(");
```

$$\mathrm{u}(x) = \frac{1}{2}\,\frac{-(x-1)\,(x-2)\,\mathrm{u}(0) - 16\,x\,(x-2)\,\mathrm{u}(1) + 70\,x\,(x-1)\,\mathrm{u}(2)}{(3\,x-1)\,(3\,x+1)\,\Gamma(x+1)}$$

Compute some values of this solution:

```
>  simplify([seq(subs(x=i,"),i=0..10)]);
```

$$[\mathrm{u}(0)=\mathrm{u}(0),\ \mathrm{u}(1)=\mathrm{u}(1),\ \mathrm{u}(2)=\mathrm{u}(2),\ \mathrm{u}(3)=-\frac{1}{480}\,\mathrm{u}(0)-\frac{1}{20}\,\mathrm{u}(1)+\frac{7}{16}\,\mathrm{u}(2),$$
$$\mathrm{u}(4)=-\frac{1}{1144}\,\mathrm{u}(0)-\frac{8}{429}\,\mathrm{u}(1)+\frac{35}{286}\,\mathrm{u}(2),\ \mathrm{u}(5)=-\frac{1}{4480}\,\mathrm{u}(0)-\frac{1}{224}\,\mathrm{u}(1)+\frac{5}{192}\,\mathrm{u}(2),$$
$$\mathrm{u}(6)=-\frac{1}{23256}\,\mathrm{u}(0)-\frac{4}{4845}\,\mathrm{u}(1)+\frac{35}{7752}\,\mathrm{u}(2),$$
$$\mathrm{u}(7)=-\frac{1}{147840}\,\mathrm{u}(0)-\frac{1}{7920}\,\mathrm{u}(1)+\frac{7}{10560}\,\mathrm{u}(2),$$
$$\mathrm{u}(8)=-\frac{1}{1104000}\,\mathrm{u}(0)-\frac{1}{60375}\,\mathrm{u}(1)+\frac{7}{82800}\,\mathrm{u}(2),$$
$$\mathrm{u}(9)=-\frac{1}{9434880}\,\mathrm{u}(0)-\frac{1}{524160}\,\mathrm{u}(1)+\frac{1}{104832}\,\mathrm{u}(2),$$
$$\mathrm{u}(10)=-\frac{1}{90619200}\,\mathrm{u}(0)-\frac{1}{5097330}\,\mathrm{u}(1)+\frac{1}{1035648}\,\mathrm{u}(2)]$$

The generating function $y(z)=\sum_{n=0}^{\infty}u(n)z^n$ is a solution of the following differential equation:

$$\frac{1}{9}\,\frac{\mathrm{y}(z)}{z^2}-\frac{1}{3}\,\frac{(-8+3\,z)\,(\frac{\partial}{\partial z}\,\mathrm{y}(z))}{z^2}-\frac{1}{3}\,\frac{(35+3\,z^2-27\,z)\,(\frac{\partial^2}{\partial z^2}\,\mathrm{y}(z))}{z^2}$$
$$+\frac{1}{9}\,\frac{(27\,z^2-135\,z+80)\,(\frac{\partial^3}{\partial z^3}\,\mathrm{y}(z))}{z^2}-\frac{(3\,z-7)\,(\frac{\partial^4}{\partial z^4}\,\mathrm{y}(z))}{z}+(\frac{\partial^5}{\partial z^5}\,\mathrm{y}(z))=0$$

A basis of the solutions of this equation can be given by $\exp(z)$, $z^{1/3}$, $z^{-1/3}$ and two more complicated solutions involving integrals.

All solutions of this difference operator L have the same type; each solution $u(x)$ is $1/\Gamma(x+1)$ times a rational function in x. However, the current algorithm (based on [13]) in Maple will still distinguish a number of different cases. The number of cases in this example can be reduced to 1 by computing the $\overline{g}_p(L)$ defined in [10] for all singularities p. In fact, whenever all solutions of a difference operator L are hypergeometric and of the same type, the number of cases in the algorithm in [10] is only 1, whereas in the method in [13] the number of cases to check can still be exponentially high. Both methods are fast on this particular operator L though, because of the fact that all roots of the leading and trailing coefficients (the coefficients of the highest and of the lowest power of τ in L) are rational numbers, which makes this particular example relatively easy. The example in the next section is much harder.

6.2

The following example was a motivation to start working on this problem. It was sent to Maple as a bug-report in the procedure `hypergeomsols` in the LREtools package. After having fixed the bug, the algorithm still did not perform satisfactory because it takes weeks to run this example.

$$\begin{aligned}
f &= a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0 \\
a_0 &= 18\,(2\,x+3)(x+2)(140\,x^3+1151\,x^2+3114\,x+2781)(x+1)^2 \\
a_1 &= -(x+2)(23660\,x^6+302879\,x^5+1581604\,x^4+4314577\,x^3 \\
&\quad +6487290\,x^2+5099454\,x+1638144) \\
a_2 &= (18380306\,x^2+13291032\,x+237304\,x^6+1637876\,x^5+4046652 \\
&\quad +14560\,x^7+6200310\,x^4+13887720\,x^3) \\
a_3 &= -4\,(140\,x^3+731\,x^2+1232\,x+678)(2\,x+7)^2(x+3)^2.
\end{aligned}$$

The current algorithm tries all combinations of all factors in $\mathbb{C}[x]$ of a_3 and all factors of a_0, that is 6912 cases. Computing with these factors requires computing with algebraic numbers; not all factors of a_0 and a_3 have rational number coefficients. These algebraic numbers make the computation slow.

The computation can be done much faster by computing the sets $\overline{g}_p(f)$ for each finite singularity $p \in \mathbb{C}/\mathbb{Z}$, as defined in [10]. The roots of a_3 are $-7/2, -3, \alpha_1, \alpha_2, \alpha_3$ where $\alpha_i \in \mathbb{C}$ are the three roots of $140x^3 + 731x^2 + 1232x + 678$. The roots of a_0 are $-3/2, -2, -1, \alpha_1 - 1, \alpha_2 - 1, \alpha_3 - 1$. Since a_0, a_1, a_2, a_3 have no poles, the finite singularities of f are the roots of $a_0 a_3$ modulo the integers, so the finite singularities are $p_1 = \alpha_1 + \mathbb{Z}, p_2 = \alpha_2 + \mathbb{Z}, p_3 = \alpha_3 + \mathbb{Z}, p_4 = 0 + \mathbb{Z}, p_5 = 1/2 + \mathbb{Z}$. Using a deformation (cf. [10]) of f one can compute the sets

$$\begin{aligned}
\overline{g}_{p_1}(f) &= \{0\} \\
\overline{g}_{p_2}(f) &= \{0\} \\
\overline{g}_{p_3}(f) &= \{0\} \\
\overline{g}_{p_4}(f) &= \{-2,-1,0,1,2\} \\
\overline{g}_{p_5}(f) &= \{-2,-1,0,1\}
\end{aligned}$$

Hence if u is a hypergeometric solution, then for its valuation growth at p where $p \in \mathbb{C}/\mathbb{Z}$ there are 5 cases if $p = p_4$, 4 cases if $p = p_5$ and only 1 case otherwise. So this leaves only $4 \cdot 5 = 20$ cases instead of the 6912 cases in the algorithm in [13]. The singularity at infinity is not taken into consideration in this comparison, although it would make the difference slightly larger because of Fuchs' relations in [10]. The fact that has an effect, much larger than this 6912 to 20 ratio, on the computation times is that in the new method the only singularities that really matter (i.e. where there is more than 1 case) are p_4 and p_5. So these 20 cases involve no algebraic extensions, and hence each of these cases is much easier than most of the 6912 cases (most of those cases do involve algebraic extensions). Given these $\overline{g}_p(f)$, the computation can be done in less than a minute, which is a big improvement over the algorithm from [13]. The result of the computation is that this example has no non-zero hypergeometric solutions.

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