FLORIDA STATE UNIVERSITY

COLLEGE OF ARTS AND SCIENCES

SOLVING THIRD AND FOURTH ORDER LINEAR DIFFERENCE EQUATIONS IN TERMS OF SECOND ORDER EQUATIONS

By

HEBA BOU KAEDBEY

A Dissertation submitted to the
Department of Mathematics
in partial fulfillment of the
requirements for the degree of
Doctor of Philisophy

2026

Heba Bou KaedBey defended this dissertation on March 2, 2026.

The members of the supervisory committee were:

Mark van Hoeij
Professor Directing Dissertation

Laura Reina
University Representative

Paolo Aluffi
Committee Member

Amod Agashe
Committee Member

Jeremy Usatine
Committee Member

The Graduate School has verified and approved the above-named committee members, and certifies that the dissertation has been approved in accordance with university requirements.

To my parents.

ACKNOWLEDGMENTS

First, I would like to express my deepest gratitude to my advisor, Professor Mark Van Hoeij. His support and guidance have been crucial in the completion of this dissertation. His insightful feedback and constructive criticism have not only refined my research but have also shaped my thinking as a researcher. Professor van Hoeij's dedication to his students is truly remarkable, and I am deeply appreciative of the countless hours he has invested in reviewing drafts, offering advice, and discussing ideas with me. His passion for the field has been a constant source of inspiration, and I am incredibly fortunate to have had the privilege of working under his guidance.

I would like to express my sincere gratitude to Dr. Man Cheung Tsui, former postdoctoral researcher at Florida State University, for his invaluable guidance and support. His feedback greatly contributed to the development of my research, particularly in the writing of our first two papers and his thoughtful input on the third paper. I truly appreciate his time, and encouragement throughout this process.

Additionally, I would like to express my appreciation to Florida State University's Department of Mathematics. For both my academic and personal development, the department has offered an encouraging environment. I am very appreciative of all the professors who have guided and taught me throughout the years, as well as the staff who have always been kind and understanding.

I would also like to thank my committee members Professor Laura Reina, Professor Paolo Aluffi, Professor Jeremy Usatine and Professor Amod Agashe for their helpful feedback on the dissertation.

To my friends, throughout this journey, your friendship and support have been a source of strength. Whether it was words of encouragement, or simply being there to listen, you have all played an essential role in helping me stay motivated and focused. I am incredibly fortunate to have such a wonderful group of friends who have made this journey not only bearable but also enjoyable.

To my parents, who have supported me from across the seas, I cannot thank you enough for your endless love, encouragement, and belief in my dreams. Despite the distance, you have always been there for me, offering comfort, and motivation whenever I needed it. Your sacrifices and

understanding of the challenges I have faced while pursuing my studies abroad have meant the world to me.

I would like to thank the NSF (grant 2007959) for partially funding my research.

TABLE OF CONTENTS

ABSTRACT

In this thesis we classify order 3 linear difference operators over $\mathbb{C}(x)$ that are solvable in terms of lower order difference operators. To prove this result, we introduce the notion of absolute irreducibility for difference modules, and classify (for arbitrary order) modules that are irreducible but not absolutely irreducible. We also classify order 4 linear difference operators over $\mathbb{C}(x)$ that are solvable in terms of lower order difference operators. We use results from difference Galois theory to prove the completeness of this classification.

We present two algorithms for computing what we call the absolute factorization of a difference operator, and improve the efficiency of one of them. We also give an algorithm for solving third order difference equations in terms of second order equations, together with applications to OEIS sequences. The latter algorithm is similar to existing algorithms for differential equations in [44, 20], except that there is an additional order one symmetric product. Lastly, we present two algorithms for the absolute irreducible cases for order four. Examples from the OEIS follow all these algorithms.

CHAPTER 1

INTRODUCTION

Computer algebra systems are often used to solve differential equations from many branches of science. Linear homogeneous differential equations with rational function coefficients are very common in mathematics, combinatorics, physics and engineering. There are numerous algorithms to find closed form solutions (solutions expressible in terms of well studied special functions, for example; Bessel, Kummer, Liouvillian, Hypergeometric functions etc.) [17, 12, 21, 13]. One way to interpret a closed form solution is that it is an expression that reduces a differential equation to a standard equation (the equation for that special function). Viewed this way, it also makes sense to reduce differential equations to any equations of lower order, regardless of whether the reduced equation belongs to a special function. Papers [44, 20] show how to reduce, whenever possible, a differential equation of order three to an equation of order two. Aspects of [44] were subsequently generalized: M. Singer [45] gave representation-theoretic criteria for solvability by lower order operators, and K. A. Nguyen [36] and Nguyen-van der Put [37] generalized and streamlined [45], and gave other solvability criteria. Also, [18, 16, 38] handle differential equations of order four. Our goal is to do this for recurrence relations, which we represent using a difference operator.

The first algorithm to compute hypergeometric solutions of recurrence relations was given by Petkovsek [39]. The first algorithm for Liouvillian solutions of recurrence relations was given by Hendriks and Singer [15, 14]. Both can be interpreted as writing solutions of a difference operator in terms of solutions of operators of order one. Our goal is to extend this to solving difference operators in terms of solutions of operators of order two.

One of the reasons for doing this is because of solvers for order 2 such as [9]. Another reason is timing. Computing 1-expressible (i.e. Liouvillian) solutions reduces to computing order-1 factors of certain auxiliary operators. To go beyond that, one needs an implementation that can find higher order factors, which now exists (RightFactors in Maple's LREtools package)

Let $D = \mathbb{C}(x)[\Phi, \Phi^{-1}]$ be the noncommutative ring of Laurent polynomials in Φ over $\mathbb{C}(x)$ with multiplication $\Phi \circ f(x) = f(x+1) \circ \Phi$. After possibly left-multiplying by Φ^k for some integer k, we can write a non-zero element of D as

$$L = a_n \Phi^n + a_{n-1} \Phi^{n-1} + \cdots + a_1 \Phi + a_0, \quad a_n, a_0 \neq 0. \tag{1.1}$$

Such L is a *difference operator* of order n. It acts on $f(x)$ by $L(f)(x) = \sum_i a_i \, f(x+i)$.

A difference operator is said to be *2-solvable* if it has a nonzero *2-expressible* solution, i.e., a solution that can be expressed[1] in terms of solutions of second order equations. The notion of a 2-solvable operator is a generalization of operators whose solutions are *Liouvillian* sequences, defined in [14].

Liouvillian solutions are expressed[1] in terms of solutions of first order equations. A property that makes Liouvillian solutions relatively easy to find is that any product of solutions of first order equations still satisfies an equation of the same order. For order 2, products can lead to higher order. Already for order 3 this adds a case, namely case (S^2) below.

Theorem 1.0.1. *Let $L \in D$ be a third order linear difference operator that is 2-solvable. Then one of the following holds:*

- *(R) (Reducible case). L admits a nontrivial factorization over $\mathbb{C}(x)$;*
- *(L) (Liouvillian case). L is gauge equivalent to $\Phi^3 + a$ for some $a \in \mathbb{C}(x)$; (See Definition 2.0.12 for a definition of gauge equivalence.)*
- *(S^2) (Symmetric square case). L is gauge equivalent to $L_2^{\circledS 2} \circledS L_1$ for some first order operator L_1 and some second order operator L_2 over $\mathbb{C}(x)$. (See Definition 2.0.14 on symmetric product.)*

Cases (R) and (L) are already handled by existing algorithms [39, 4, 14]. For case (S^2) we develop an algorithm in Chapter 5. To analyze these cases, we introduce representation-theoretic ideas that contextualize the results of Hendriks-Singer [14]. We will formulate results in terms of difference modules to simplify proofs. Two natural operations on sequences—sectioning and interlacing—readily correspond to restriction and induction. This was hinted at by the work of K. Amano [3, Section 3.7] which also generalized [14] to Artinian simple module algebras.

In this thesis we will prove that every 2-solvable order 4 operator $L \in D$ falls in one of these four cases:

[1] using difference ring operations, indefinite summation, and interlacing; see Chapter 3 for more details

1. L is reducible in D, or
2. L is irreducible but not absolutely irreducible, or
3. $(D/DL)\downarrow_2^1$ is isomorphic to a tensor product of two 2-dimensional modules, or
4. L is gauge equivalent to some $L_2^{\circledS 3} \circledS (\Phi - r)$, where L_2 has order 2.

To find 2-expressible solutions for any 2-solvable recurrences of order 4, we need an algorithm for each case. For Case (1) see [50, 4, 25]. For Case (2), absolute factorization, see Chapter 4. For Case (3), we develop an algorithm in Chapter 6 that reduces L (or its "section operator" $L\downarrow_2^1$) to a symmetric product of two order two operators. Chapter 6 gives an algorithm for Case (4) as well.

The thesis is organized as follows. Chapter 2 recalls relevant facts, and give results about determinants of difference operators and modules that are needed to makes the algorithms in Chapters 4 and 6 more efficient. Chapter 3 divides as follow: Section 3.1 formally defines 2-expressible sequences and relates them to Eulerian groups. This section also characterizes operators whose solutions are 2-expressible. The symmetric power construction as recalled in Section 3.2 gives us a criterion (Proposition 3.2.1) to check for Case (S^2) of order 3 classification theorem. Section 3.3 defines induced and restricted difference modules. Subsection 3.3.1 gives examples of how these operations help with simplifying difference operators. Section 3.4 describes theoretical results on induction and restriction, most notably Theorem 3.4.4, which characterizes the irreducible but not absolutely irreducible modules. In Section 3.5, we prove our main result for order 3, Theorem 3.5.1. In Section 3.6, we state the classification for order 4 theorem. In Section 3.7 we prove some lemmas and theorems that ensure the completeness of the list of cases in Theorem 3.6.2. Chapter 4 gives two algorithms for *absolute factorization*. We use the results of determinants in Chapter 2 to improve absolute factorization by reducing the number of combinations that the factorizer needs to check, using Theorem 4.3.1 to discard unnecessary combinations. Section 4.2 explains in what situations this is beneficial (in what situations the number of cases will be high), and we include an example that shows the computational time and number of cases before and after the improvement. As illustrated in Example 4.4, this covers one of the cases in the classification of 2-solvable order 4 equations. Moreover, the orders of the resulting factors in this factorization through the notion of *absolute order* in Section 4.5 are described. Chapter 5 covers case (S^2): given $L \in D$ of order 3, decide if case (S^2) holds, and if so, return L_2, $\Phi - r$, and the gauge transformation. We give an example from the OEIS where our algorithm produces an output that proves a conjecture from

Z.-W. Sun. In Chapter 6 we give algorithms for Cases (3) and (4) of the order 4 classification, and examples from the OEIS. Section 6.2 also gives a formula for the determinant for the algorithm of Case (4), without this, it would be impractically slow. Implementations and experiments are available at [24].

The thesis ends with seven appendices. Appendix A gives more explanation for the proof of Proposition 3.2.1. Appendix B completes the proof of 3.2.1. Appendix C covers a difference analogue of Mackey's formula that go beyond the results discussed in Section 3.4. Appendix D explains how our notion of absolute irreducibility aligns with other common usages of the term. Appendix E proves [42, Exercise 2.38 (4)] which states that the tensor product of two completely reducible modules is again completely reducible. Appendix F states some results from various papers, that show that $\mathrm{Sp}_4(\mathbb{C})$ and $\mathrm{SL}_4(\mathbb{C})$ are non Eulerian. Finally, Appendix G gives an explanation for the exterior square of a tensor product.

1.1 Sources and Prior Publications

This dissertation is based in part on previously published joint work with Professor Mark Van Hoeij and Dr. Man Cheung Tsui.

Chapter 2 reproduces results from [7], published in the Journal of Symbolic Computation (2025), as well as results from [8], publioshed in the Journal of Symbolic Computation (2026).

Chapter 3 contains results from [7], together with new results that have not yet appeared elsewhere.

Chapter 4 reproduces results from [6], published in the Proceedings of the 2024 International Symposium on Symbolic and Algebraic Computation, as well as results from [8].

Chapter 5 reproduces results from [6].

Chapter 6 reproduces results from [8].

1.2 Future Work

As future work, we expect that (from private communication with M. van der Put) an adaptation of [36] and [37] will determine when difference modules are solvable in terms of lower-dimensional difference modules. This could simplify the proofs given here and describe d-solvability in general.

Also, there are many recurrences that produce an integer sequence despite the fact that at every stage we divide by the leading coefficient. When this happens it seems too consistent to be a coincidence; in sense that there should be some explanation (some kind of formula) that shows why it is an integer sequence. That is one of the motivations for looking for algorithms that reduce formulas. I plan to investigate algorithmic methods that explain why certain recurrences yield integer sequences despite divisions, building on patterns observed in some OEIS examples stated in the thesis.

CHAPTER 2

PRELIMINARIES

This chapter is reproduced from joint work [7] and [8].

This section introduces some notations and facts about difference operators and modules that we will use. The standard facts we use are from [14, 41].

Consider the ring $\mathcal{S} = \mathbb{C}^{\mathbb{N}}/\sim$ under the equivalence relation $u \sim v$ if $(u-v)(x) \neq 0$ for finitely many x. An element of $\mathcal{S}$ is called a *sequence* and represented as a function $u : \mathbb{N} \to \mathbb{C}$ or a list $(u(0), u(1), \dots)$. The set $\mathcal{S}$ is a difference ring under component-wise addition and multiplication, and $\phi(u(x)) = u(x+1)$. Since rational functions have finitely many poles, $\mathbb{C}(x)$ embeds into $\mathcal{S}$ by evaluation at $\mathbb{N}$, making $\mathcal{S}$ a $\mathbb{C}(x)$-algebra as well as a (left) D-module where $D = \mathbb{C}(x)[\Phi, \Phi^{-1}]$, and where Φ acts as ϕ.

Definition 2.0.1. *If R is a $D-$module then $V_R(L) = \{u \in R \mid Lu = 0\}$.*

If R is not specified then we take R to be $\mathcal{S}$:

Definition 2.0.2. *The* solution space *$V(L)$ of an operator L is the set $\{u \in \mathcal{S} \mid Lu = 0\}$.*

Remark 2.0.3. *This is a $\mathbb{C}$-vector space of dimension* order(L) *by [40, Theorem 8.2.1].*

We can define some notions about operators in terms of D-modules.

If u is an element of a D-module, the *minimal operator* of u, denoted by $\mathrm{MinOp}(u, D)$, is a generator of the left ideal $\{L \in D \mid L(u) = 0\}$ of D. To make it unique, we require it to be monic in $\mathbb{C}(x)[\Phi]$. Given $L \in D$ and $u \in V(L)$, the Maple command `MinimalRecurrence` computes $\mathrm{MinOp}(u, D)$.

Let M be a D-module. Then $u \in M$ is a *cyclic vector* if $Du = M$. We call M *irreducible* if every nonzero element of M is cyclic; equivalently, the minimal operator of each element of M is irreducible of order $\dim(M)$.

The *solution space* in R of a D-module M is the set

$$V_R(M) \coloneqq \mathrm{Hom}_D(M, R).$$

An *R-point* of M is an element of $V_R(M)$. If m is a cyclic vector of M and L is the[1] minimal operator of m, then the evaluation map

$$V_R(M) \to V_R(L): \quad s \mapsto s(m)$$

defines a bijection between the R-points of M and the solutions of L in R. We say $V_R(M)$ is a *full solution space* if its $\mathbb{C}$-dimension is $\dim_{\mathbb{C}(x)} M$. This will be the case if we choose $R = \mathcal{S}$ (Remark 2.0.3).

Remark 2.0.4. *Let L be the minimal operator of u. We have $Du \cong D/DL$.*

Lemma 2.0.5. *If L_2 is a right hand factor of L then $V(L_2) \subseteq V(L)$.*

Proof. If $u \in V(L_2)$ then $L_2(u) = 0$. Then, $L(u) = L_1 L_2(u) = L_1(0) = 0$. □

For $L_1, L_2 \in D$, the operators $\mathrm{GCRD}(L_1, L_2)$ and $\mathrm{LCLM}(L_1, L_2)$ are defined as generators of the left ideals $DL_1 + DL_2$ and $DL_1 \cap DL_2$ of D, respectively. Then the following theorem holds.

To make those generators unique we require that they are in $\mathbb{C}(x)[\Phi]$, monic, and with non-zero Φ^0 coefficient.

Theorem 2.0.6.

$$V(\mathrm{GCRD}(L_1, L_2)) = V(L_1) \cap V(L_2) \tag{2.1}$$

and

$$V(\mathrm{LCLM}(L_1, L_2)) = V(L_1) + V(L_2) \tag{2.2}$$

Proof. Let $G = \mathrm{GCRD}(L_1, L_2)$. $V(G) \subseteq V(L_1) \cap V(L_2)$ by Lemma 2.0.5. Conversely, if $u \in V(L_1) \cap V(L_2)$ then $u \in V(L)$ for any $L \in DL_1 + DL_2$. But, $G \in DL_1 + DL_2$, by the definition of the GCRD, hence equation 2.1

Let $L = \mathrm{LCLM}(L_1, L_2)$, and $\widetilde{L_i} \in D$ such that $L_i = \widetilde{L_i} G$. $V(L_1) + V(L_2) \subseteq V(L)$ by Lemma 2.0.5. Conversely, let $d = \mathrm{order}(G)$ and $N_i = \mathrm{order}(L_i)$. Let $L = \sum_{i=0}^{N_1+N_2} a_i \Phi^i$, where a_i's are unknowns.

[1] Minimal operators and LCLM's are unique up to left multiplication by units in D.

Dividing L by L_1 and setting remainder to 0, we get N_2 equations and dividing L by L_2 and setting remainder to 0, we get N_1 equations. Thus, we have $N_1 + N_2$ equations, but, we have $N_1 + N_2 + 1$ unknowns. So, there exists a common left multiple of order at most $N_1 + N_2$. Thus,

$$\text{order}(\text{LCLM}(L_1, L_2)) \leq \text{order}(L_1) + \text{order}(L_2), \tag{2.3}$$

holds for all $L_1, L_2 \in D$.

Let $\widetilde{L} = \text{LCLM}(\widetilde{L_1}, \widetilde{L_2})$. We have:

$$\text{order}(\widetilde{L}) \leq \widetilde{N_1} + \widetilde{N_2} \tag{2.4}$$

and

$$\text{order}(\widetilde{L_i}) = \widetilde{N_i} = N_i - d \tag{2.5}$$

Let $L = \widetilde{L}G$. Using equations 2.4 and 2.5 we get

$$\text{order}(L) \leq \text{order}(L_1) + \text{order}(L_2) - \text{order}(G).$$

$\dim(V(L)) = \text{order}(L) \leq \text{order}(L_1) + \text{order}(L_2) - \text{order}(G) = \dim(V(L_1) + V(L_2))$. But, we have proved that $V(L_1) + V(L_2) \subseteq V(L)$ and hence equation (2.2). □

Theorem 2.0.7. *$L_1 \in DL_2$ if and only if $V(L_2) \subseteq V(L_1)$.*

Proof.

$\Rightarrow$ This is Lemma 2.0.5.

$\Leftarrow$ We have $V(L_2) \subseteq V(L_1)$, i.e $V(L_1) \cap V(L_2) = V(L_2)$. Then $\text{GCRD}(L_1, L_2) = L_2$ by part 1 of Theorem 2.0.6, so L_2 right divides L_1. □

Remark 2.0.8. *Since L vanishes on $V(L)$, any $G + DL \in D/DL$ gives a well defined map $G : V(L) \to \mathcal{S}$. This is the zero map if and only if $G + DL = 0 + DL$.*

Lemma 2.0.9 ([42, Proposition 2.13] or [46, Lemma 2.5])**.** *Let $L_1, L_2 \in D$. Then the map*

$$\Psi : \text{Hom}_D(D/DL_1, D/DL_2) \longrightarrow \text{Hom}_{\mathbb{C}}(V(L_2), V(L_1))$$
$$f \mapsto f(\bar{1})$$

is injective, where $\bar{1} = 1 + DL_1$.

Proof. By Remark 2.0.8, $f(\bar{1}) \in D/DL_2$ gives a map $f(\bar{1}) : V(L_2) \to \mathcal{S}$ which vanishes if and only if $f(\bar{1})$ is zero in D/DL_2. But, since L_1 annihilates $\bar{1}$ and thus $f(\bar{1})$ since f is a $D-$Homomorphism. It follows that $f(\bar{1}) : V(L_2) \to V(L_1)$. Injectivity follows from the fact that if $f(\bar{1})$ is zero in D/DL_2, then f is zero since $\bar{1}$ generates D/DL_1. □

$\mathrm{Hom}_D(D/DL_1, D/DL_2)$ can be computed [4, 23]. An implementation will be included in Maple 2026, to be released this year. It can be used to factor operators when the endomorphism ring is non-trivial:

Lemma 2.0.10 ([46, Corollary 2.7])**.** *Let $M = D/DL$. If $\mathrm{End}_D(M) \neq \mathbb{C}$ then M is reducible.*

Proof. If $c \in \mathbb{C}$ then c commutes with any element of D and hence $c \in \mathrm{End}_D(M)$. Now suppose that $f \in \mathrm{End}_D(M) \setminus \mathbb{C}$. Let Ψ be as in Lemma 2.0.9. Let $P(T) \in \mathbb{C}[T]$ be the minimal polynomial of $\Psi(f) \in \mathrm{End}_{\mathbb{C}}(V(L)) \cong \mathrm{Mat}_{n,n}(\mathbb{C})$. (The minimal polynomial of an n by n matrix divides the characteristic polynomial).

Since Ψ is injective (Lemma 2.0.9) then $P(T)$ is also the minimal polynomial of f, and since f is not constant, the degree of P is more than 1. Now write $P(T) = P_1(T)P_2(T)$ for some non-constant $P_1, P_2 \in \mathbb{C}[T]$. Then $P_1(f), P_2(f) \in \mathrm{End}_D(M)$ are non-zero and their composition $P_1(f)P_2(f)$ is zero. Hence $P_1(f)$ has a non-trivial kernel and thus M is reducible. □

Lemma 2.0.11. *Let $L_1, L_2 \in D$ have the same order. The following statements are equivalent:*

1. *$D/DL_1 \cong D/DL_2$ as D-modules.*
2. *There exists $G \in D$ such that $G(V(L_2)) = V(L_1)$.*
3. *There exists $G \in D$ (same as in (2)), and such that $L_1 G \in DL_2$ and $\mathrm{GCRD}(G, L_2) = 1$.*
4. *There exists a D-module with cyclic vectors u_1, u_2 such that $L_1 = \mathrm{MinOp}(u_1, D)$ and $L_2 = \mathrm{MinOp}(u_2, D)$.*

Proof. 1. $\Rightarrow$ 2.: Let $\psi : D/DL_1 \to D/DL_2$ be the given isomorphism. Let $\bar{1} := 1 + DL_1$, and let $G \in D$ be a lift of $\psi(\bar{1}) \in D/DL_2$. This completely determines the isomorphism because G is a generator. First, we need to check that $G : V(L_2) \to V(L_1)$ is well-defined. Suppose that $G' \in D$ satisfies $\overline{G}' = \overline{G} \in D/DL_2$. Then $G' = G + QL_2$ for some $Q \in D$. We want to show that $G'(f) = G(f)$ for all $f \in V(L_2)$. Note that $G'(f) = (G + QL_2)(f) = G(f)$. Thus, G is well-defined.

We have $L_1 \cdot \overline{1} = 0$ in D/DL_1, then $L_1 \cdot \psi(\overline{1}) = L_1 \cdot G = 0$ in D/DL_2 when $L_1 G \in DL_2$. So, $L_1 G = QL_2$, for some $Q \in D$.

Now, take $u_2 \in V(L_2)$. We get $(QL_2)(u_2) = (L_1 G)(u_2) = 0$. Therefore, $G(u_2) \in V(L_1)$.

2. $\Rightarrow$ 3.: Let G be as in 2. Then, $L_1 G \in DL_2$ as before.

Given that $\operatorname{order}(L_1) = \operatorname{order}(L_2)$. By Theorem 2.0.3, this is the same as $\dim V(L_1) = \dim V(L_2)$. From 2. we get that $\dim GV(L_2) = \dim V(L_2)$, which is equivalent to having $\operatorname{GCRD}(G, L_2) = 1$.

3. $\Rightarrow$ 4.: Take $M = D/DL_2$, with $m_1 = G + DL_2$ and $m_2 = 1 + DL_2$.

4. $\Rightarrow$ 1.: Follows from Remark 2.0.4. □

Definition 2.0.12. *If any statement in Lemma 2.0.11 holds, we say that L_1 is* gauge equivalent *to L_2, and G is a* gauge transformation *from L_2 to L_1. Moreover, $G : V(L_2) \to V(L_1)$ is a bijection.*

Remark 2.0.13. *Computing the inverse of G. Since $\operatorname{GCRD}(G, L_2) = 1$ in D, the extended Euclidean algorithm finds $\widetilde{G}, R \in D$ with $\widetilde{G}G + RL_2 = 1$. Thus $\widetilde{G}G \equiv 1 \pmod{L_2}$ and $\widetilde{G}G$ is the identity map on $V(L_2)$. Therefore $\widetilde{G} : V(L_1) \to V(L_2)$ is the inverse of G.*

Definition 2.0.14. *Let $L, L' \in D$. The* symmetric product *$L \circledS L'$ is the minimal operator of $1 \otimes 1 \in (D/DL) \otimes (D/DL')$.*

Note that $L^{\circledS 2} = L \circledS L$. If L_1 has order 1 and λ is a non-zero solution of L_1 then $V(L_1 \circledS L) = \lambda \cdot V(L)$. Like [41, Corollary 2.19], we have

$$V_{\mathcal{S}}(L \circledS L') = \operatorname{span}_{\mathbb{C}} \left\{ u \cdot v \,\middle|\, u \in V_{\mathcal{S}}(L), v \in V_{\mathcal{S}}(L') \right\}. \tag{2.6}$$

Definition 2.0.15. *Operators $L, L' \in D$ are* projectively equivalent *if there exists $L_1 \in D$ of order 1 for which $L_1 \circledS L$ is gauge equivalent to L'. A* projective transformation *is a pair (L_1, G), where G is a gauge transformation from $L_1 \circledS L$ to L'. This expresses solutions of L' in terms of solutions of L_1 and L. In particular, if L is 2-solvable, so is L'.*

Definition 2.0.16. *(Matrix for M). If M is a D-module then we say that A* is a matrix for M *if it* expresses the action *of ϕ on a basis of M. If $b_1, \ldots, b_n$ is that basis, then*

$$\begin{pmatrix} \phi(b_1) \\ \vdots \\ \phi(b_n) \end{pmatrix} = A \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}.$$

Remark 2.0.17. *(Change of basis). If A is a matrix for M then so is $\phi(P)AP^{-1}$ for any $P \in \mathrm{GL}_n(\mathbb{C}(x))$.*

Definition 2.0.18. *(Determinant). If M is a D-module of dimension n, then* $\det(M)$ *is the 1-dimensional D-module $\bigwedge^n M$.*

We typically represent $\det(M)$ with a rational function as follows.

Definition 2.0.19. *If M_1 is a 1-dimensional D-module and $r \in \mathbb{C}(x) - \{0\}$ then we write $M_1 \sim r$ when $M_1 \cong D/D(\Phi - r)$, i.e., when (r) is a 1×1 matrix for M_1. If $r_1, r_2 \in \mathbb{C}(x) - \{0\}$ then we write $r_1 \sim r_2$ when $D/D(\Phi - r_1) \cong D/D(\Phi - r_2)$ which by Remark 2.0.17 is equivalent to $r_1/r_2 = \phi(P)/P$ for some $P \in \mathbb{C}(x) - \{0\}$.*

Definition 2.0.20. *(Companion matrix and determinant of L). The* companion matrix *of L from equation (1.1) is:*

$$C_L := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & 1 \\ -a'_0 & -a'_1 & -a'_2 & \cdots & -a'_{n-1} \end{pmatrix}$$

where $a'_i = a_i/a_n$ (recall that a_0 and a_n are non-zero). It expresses the action of ϕ on the basis $1, \ldots, \phi^{n-1}$ of D/DL. Because of Lemma 2.1.1 below we define the determinant *of L as* $\det(L) := \det(C_L) = (-1)^n a'_0 = (-1)^n a_0/a_n$.

2.1 Symmetric and Exterior Powers

Let k be a field. Let V be a k-vector space with basis $B := \{b_1, \ldots, b_n\}$. By viewing $b_1, \ldots, b_n$ as variables, we obtain a polynomial ring $k[b_1, \ldots, b_n]$. We identify $\mathrm{Sym}^d(V)$ with the set of homogeneous polynomials of degree d. Let $B_d := \{b_1^d, b_1^{d-1}b_2, \ldots, b_n^d\}$ be a basis.

If A is the matrix w.r.t. B of a linear map $h : V \to V$, then the *symmetric power matrix* $\mathrm{Sym}^d(A)$ is the matrix w.r.t. B_d of the induced linear map $\mathrm{Sym}^d(V) \to \mathrm{Sym}^d(V)$. For elementary matrices it is easy to check that

$$\det(\mathrm{Sym}^d(A)) = \det(A)^m \quad \text{where} \quad m = \binom{n+d-1}{n}. \tag{2.7}$$

Then (2.7) holds in general because both det and Sym^d preserve products. Likewise, h induces a linear map $\bigwedge^d(V) \to \bigwedge^d(V)$. We denote its matrix w.r.t. basis $\{b_{i_1} \wedge \cdots \wedge b_{i_d} \mid i_1 < \cdots < i_d\}$ as $\bigwedge^d(A)$.

$$\det(\bigwedge^d(A)) = \det(A)^k \quad \text{where} \quad k = \binom{n-1}{d-1}. \tag{2.8}$$

If M and N are D-modules, then the *tensor product* $M \otimes N$ is a D-module under $\Phi(m \otimes n) := \Phi(m) \otimes \Phi(n)$. The D-module structure of symmetric and exterior powers of M is defined in a similar way.

Lemma 2.1.1. *Let M be an n-dimensional D-module and let A be a matrix for M. Then* $\det(M) \sim \det(A)$, $\det(\mathrm{Sym}^d(M)) \sim \det(A)^m$, *and* $\det(\bigwedge^d(M)) \sim \det(A)^k$ *with m, k as in equations (2.7) and (2.8). If $M \cong D/DL$ then* $\det(M) \sim \det(C_L) = \det(L)$.

Proof. Let $b_1, \ldots, b_n$ be a basis of M and $b := b_1 \wedge \ldots \wedge b_n$ a basis of $\det(M)$. Now $\phi(b) = \phi(b_1) \wedge \ldots \wedge \phi(b_n) = \det(A)b$, so $\det(M) \sim \det(A)$. The formulas for $\mathrm{Sym}^d(M)$, $\bigwedge^d(M)$ and D/DL hold because $\mathrm{Sym}^d(A)$, $\bigwedge^d(A)$ and C_L are matrices for these modules. □

2.2 Symmetric and Exterior Powers of Operators

If $L \in D$ has order n then $b_1, \ldots, b_n := \Phi^0, \ldots, \Phi^{n-1}$ is a basis of $M := D/DL$. Let $L^{\circledS d}$ be the minimal operator of $b_1^d \in \mathrm{Sym}^d(M)$. If b_1^d is a cyclic vector (if it generates $\mathrm{Sym}^d(M)$) then we say the order of $L^{\circledS d}$ is *as expected*, in which case

$$\mathrm{order}(L^{\circledS d}) = \dim(\mathrm{Sym}^d(M)) = \binom{n+d-1}{d}$$

and

$$\det(L^{\circledS d}) \sim \det(L)^{\binom{n+d-1}{n}}$$

by Lemma 2.1.1. If b_1^d is not a cyclic vector, then we say that the order of $L^{\circledS d}$ is *lower than expected*.

Likewise, let $\bigwedge^d(L)$ be the minimal operator of $b_1 \wedge \ldots \wedge b_d \in \bigwedge^d(M)$. If this is a cyclic vector then we say that the order is as expected, in which case

$$\mathrm{order}(\bigwedge^d(L)) = \dim(\bigwedge^d(M)) = \binom{n}{d}$$

and

$$\det(\bigwedge^d(L)) \sim \det(L)^{\binom{n-1}{d-1}}.$$

Lemma 2.2.1. *[49, Zehfuss determinant] Let M_1, M_2 be n and m-dimensional D-modules respectively and let A, B be the matrices for M_1, M_2 respectively. Then,*

$$\det(A \otimes B) = \det(A)^m \det(B)^n.$$

2.3 Difference Galois Theory

In this thesis $F = \mathbb{C}(x)$, but difference Galois theory also works over a more general difference field, provided that its field of constants is algebraically closed. Let $D = F[\Phi]$ and let M be a D-module with an associated matrix difference equation $\phi(Y) = AY$, $A \in \mathrm{GL}_n(F)$.

Definition 2.3.1. *A* Picard-Vessiot ring *for M over F is a D-algebra R satisfying the following conditions.*

1. $\{f \in R \mid \phi(f) = f\} = \mathbb{C}$.
2. *The only ideals I of R that satisfy $\phi(I) \subseteq I$ are 0 and R.*
3. $R = F[z_{ij}, \det(Z)^{-1} \mid 1 \leq i, j \leq n]$ *for some $Z = (z_{ij}) \in \mathrm{GL}_n(R)$ satisfying $\phi(Z) = AZ$.*

A Picard-Vessiot ring for M exists and is unique up to an isomorphism of D-algebras [41, Proposition 1.9]. The definition only mentioned A, but not M, in item 3, however, if A_1, A_2 are matrices for the same module (Definition 2.0.16) then their Picard-Vessiot rings are the same [41, page 24]. If $L \in D$ then its Picard-Vessiot ring is the Picard-Vessiot ring of D/DL.

Let R be a Picard-Vessiot ring of M. The *difference Galois group* of R/F is the group $\mathrm{Gal}(R/F)$ of D-algebra automorphisms of R. Let $G = \mathrm{Gal}(R/F)$. By [41, Theorem 1.13], G is a linear algebraic group over C. For $\sigma \in G$ and $s \in V_R(M)$, $(\sigma, s) \mapsto \sigma \circ s$ defines a G-action on $V_R(M)$ and induces an injective homomorphism $G \to \mathrm{GL}(V_R(M))$ of algebraic groups.

The total ring of fractions K of a Picard-Vessiot ring of M is called a *total Picard-Vessiot ring* of M. In place of M, we often refer to the (total) Picard-Vessiot ring of an operator L or matrix equation associated to M. We also let $\mathrm{Gal}(K/F)$, $\mathrm{Gal}(M/F)$, $\mathrm{Gal}(M)$, and $\mathrm{Gal}(L)$ refer to $\mathrm{Gal}(R/F)$.

Proposition 2.3.2 ([14, Corollary A.7])**.** *Let M be a D-module that has a cyclic vector and a total Picard-Vessiot ring K. The map $N \mapsto V_K(N)$ defines a bijection between the D-module quotients of M and the $\mathrm{Gal}(M)$-invariant subspaces of $V_K(M)$. Moreover, $V_K(N)$ is a full solution space of N for each quotient module N of M.*

Remark 2.3.3. *In terms of operators $L \in D$ this means that the $\mathrm{Gal}(L)$-submodules of $V(L)$ (i.e. $\mathrm{Gal}(L)$-invariant subspaces) are the solution spaces of right-factors of L. Irreducible $\mathrm{Gal}(L)$-submodules of $V(L)$ are the solution spaces of the irreducible right-factors of L.*

Proposition 2.3.4. *Let K/F be a total Picard-Vessiot extension with difference Galois group G. Let $\mathcal{F}$ be the set of $D-$subalgebras E of K such that every non zero-divisor of E is a unit of E. Let $\mathcal{G}$ be the set of algebraic subgroups of G.*

1. *The maps $\mathcal{F} \to \mathcal{G} \colon E \mapsto \mathrm{Gal}(K/E)$ and $\mathcal{G} \to \mathcal{F} \colon H \mapsto K^H$ are inverses.*
2. *If E/F is a total Picard-Vessiot extension with $E \subseteq K$, then $\mathrm{Gal}(K/E) \trianglelefteq G$.*

Proof. 1 is [41, Theorem 1.29], and 2 follows from the computation

$$\sigma \,\mathrm{Gal}(K/E)\sigma^{-1} = \mathrm{Gal}(K/\sigma(E)) = \mathrm{Gal}(K/E)$$

which holds for all $\sigma \in G$ since E/F is a total Picard-Vessiot extension. □

Lemma 2.3.5. *Let $L \in D$, let K be its Picard-Vessiot ring and let $y_1, \ldots, y_n \in K$ be a basis of solutions.*

1. *There exists an operator $R_i \in K[\Phi]$ such that $R_i(y_j)$ is 1 if $i = j$ and 0 if $i \neq j$.*
2. *If $R = a_{n-1}\Phi^{n-1} + \cdots + a_0\Phi^0 \in K[\Phi]$ and $R(y_i) = 0$ for all i, then $R = 0$.*

Proof. Since $M = D/DL$, the matrix A in Definition 2.3.1 is the companion matrix of L, and the matrix Z is $(\phi^i(y_j))$ with $i \in \{0, \ldots, n-1\}$ and $j \in \{1, \ldots, n\}$. Let $\tilde{R}_1 \in K[\Phi]$ be the determinant of

$$\begin{pmatrix} \Phi^0 & y_2 & \cdots & y_n \\ \Phi^1 & \phi(y_2) & \cdots & \phi(y_n) \\ \vdots & \vdots & \ddots & \vdots \\ \Phi^{n-1} & \phi^{n-1}(y_2) & \cdots & \phi^{n-1}(y_n) \end{pmatrix}.$$

By construction, $\tilde{R}_1(y_j) = 0$ for $j = 2, \ldots, n$, and $\tilde{R}_1(y_1) = \det(Z)$. Now take $R_1 := \det(Z)^{-1}\tilde{R}_1$. Note that $\det(Z)^{-1}$ is in K, see item 3 in Definition 2.3.1.

As for the second claim, if R maps $y_1, \ldots, y_n$ to zero, then $(a_0, \ldots, a_{n-1})Z = 0$. But Z is invertible, and hence $R = 0$. □

Definition 2.3.6. *If G acts on two finite dimensional vector spaces V_1 and V_2, then we say that V_1, V_2 are projectively equivalent G-modules if there are bases $y_1, \ldots, y_n$ of V_1 and $z_1, \ldots, z_n$ on V_2, such that for any $g \in G$, if matrix A_1 describes the action of g on $y_1, \ldots, y_n$, and matrix A_2 describes the action of g on $z_1, \ldots, z_n$, then $A_1 = c_g A_2$ for some $c_g \in \mathbb{C}^*$.*

Proposition 2.3.7. *Let $L, L' \in D$ and let K be a Picard-Vessiot extension that contains full solution spaces of L and L', and let G be the difference Galois group of K. Then*

1. *L is gauge equivalent (Definition 2.0.12) to L' if and only if $V(L)$ and $V(L')$ are isomorphic G-modules.*
2. *If L is projectively equivalent to L', and K contains the solutions of the order-1 operator L_1 from Definition 2.0.15 then $V(L)$ and $V(L')$ are projectively equivalent G-modules. Conversely, if $V(L)$ and $V(L')$ are projectively equivalent G-modules then L is projectively equivalent to L'.*

Proof. If L, L' are gauge-equivalent, then there is an operator $R \in D = F[\Phi]$ that maps $V(L)$ bijectively to $V(L')$, see Lemma 2.0.11 (where R was called G). The group G acts trivially on the coefficients of R because they are in F. The action of G also commutes with Φ and with ring operations, so it commutes with R, and hence $R : V(L) \to V(L')$ is an isomorphism of G-modules.

Conversely, suppose that $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ are bases of $V(L)$ and $V(L')$ such that the action of G on $y_1, \ldots, y_n$ is the same as that on $z_1, \ldots, z_n$. Let $R = z_1 R_1 + \cdots + z_n R_n$ with R_i as in Lemma 2.3.5. Then R maps y_i to z_i so it commutes with the action of G; if $g \in G$, then $g(R(x_i)) = R(g(x_i))$. But $g(R(x_i)) = R^g(g(x_i))$ where R^g denotes the image of R after applying g to its coefficients. Hence $R - R^g$ maps $V(L_1)$ to zero, but then $R - R^g = 0$ by lemma 2.3.5. This means that the coefficients of R are invariant under G, which by the Galois correspondence implies that $R \in D$. Hence R is a gauge transformation from $V(L)$ to $V(L')$.

If L is projectively equivalent to L', then there is a bijection from $V(L)$ to $V(L')$ of the form $y \mapsto R(\lambda y)$ with $R \in D$ and λ a solution of an operator L_1 of order 1 (Definition 2.0.15). If $g \in G$

then $g(\lambda) \in V(L_1) = \mathbb{C}\lambda$, so $g(\lambda) = c_g\lambda$ for some $c_g \in \mathbb{C}^*$, and $R^g = R$. Thus, the action of g on $V(L')$ only differs from the action on $V(L)$ by a scalar factor c_g.

Conversely, suppose that $y_1, \ldots, y_n$ and $z_1, \ldots, z_n$ are bases of $V(L)$ and $V(L')$ such that for any $g \in G$, the action of g on $y_1, \ldots, y_n$ is the same, up to a factor $c_g \in \mathbb{C}^*$, as that on $z_1, \ldots, z_n$. Then, with $R = z_1R_1 + \cdots + z_nR_n$, we have $R^g = c_gR$ for every $g \in G$. If $u \in K$ is the leading coefficient of R, then $g(u) = c_gu$, and so $\mathbb{C}u$ is G-invariant. Under the Galois correspondence, a 1-dimensional G-invariant space $\mathbb{C}u$ equals $V(L_1)$ for some operator L_1 of order 1. Now $Ru^{-1} \in K[\Phi]$ is G-invariant, hence in D, and thus (L_1, Ru^{-1}) is a projective transformation from L to L'. □

Remark 2.3.8. *In the same way one can also show that the image of Ψ from Lemma 2.0.9 is precisely the set of linear maps from $V(L_2)$ to $V(L_1)$ that are invariant under the difference Galois group.*

Remark 2.3.9. *Maple 2026 which will be released this year contains a command Homomorphisms that can compute a gauge equivalence, or a projective equivalence, if it exists. The latter is reduced to the former as follows: If L is projectively equivalent to L' then $\det(L_1)^n \det(L) \sim \det(L')$ with L_1 as in Definition 2.0.15, and $n = \operatorname{order}(L)$. Hence, $\det(L_1)$ can be determined, up $\sim$, by taking the n'th roots of $\det(L')/\det(L)$ reduced mod $\sim$. (These n'th roots differ by roots of unity. Homomorphisms are computed for each n'th root.)*

CHAPTER 3

2-SOLVABLE DIFFERENCE OPERATORS

Part of this chapter is reproduced from joint work [7] and part contains new results that have not appeared elsewhere.

3.1 2-Expressible Sequences

As we mentioned in the beginning of Chapter 2, $\mathbb{C}(x)$ embeds into $\mathcal{S} = \mathbb{C}^{\mathbb{N}}/\sim$. This difference ring contains a basis of solutions of every $L \in D$, so any Picard Vessiot extension of $\mathbb{C}(x)$ can be embedded in $\mathcal{S}$. The same is also true for $\mathbb{C}^{-\mathbb{N}}/\sim$ or $\mathbb{C}^{c-\mathbb{N}}/\sim$ or $\mathbb{C}^{c+\mathbb{N}}/\sim$, where $c - \mathbb{N} = \{c, c-1, c-2, c-3, \ldots\}$ and $c + \mathbb{N} = \{c, c+1, c+2, c+3, \ldots\}$.

This section describes a generalization of Liouvillian sequences that we will call 2-expressible sequences. Similar to how [14] classified difference operators with Liouvillian solutions, our goal is to classify operators with 2-expressible solutions, with some partial progress described in Section 3.3.1 and a definitive result in Theorem 3.5.1. For this, we study the relation between difference operators, 2-expressible solutions, and the Galois group.

We construct new sequences from old ones as follows. For $s \in \mathcal{S}$,

$$\Sigma s \coloneqq \{t \in \mathcal{S} \mid (\phi - 1)t = s\}$$

is the set of *indefinite sums* of s. The *interlacing* of $s_0, s_1, \ldots, s_{m-1} \in \mathcal{S}$ is the sequence t defined by $t(mn + i) = s_i(n)$ for $n \geq 0$ and $0 \leq i < m$. The *m-interlacing* of s *with zeros* is defined as the interlacing of s with $(m-1)$-many zero sequences.

Remark 3.1.1.

1. *The interlacing of $s_0, s_1, \ldots, s_{m-1}$ is the sum $\sum \phi^{-i}(t_i)$ where t_i is the m-interlacing of s_i with zeros.*

2. *If s satisfies an operator $L(x, \Phi) = \sum_i f_i(x)\Phi^i$, then the m-interlacing of s with zeros satisfies the operator $L(x/m, \Phi^m) = \sum_i f_i(x/m)\Phi^{im}$.*

In [14, Definition 3.3] the ring of *Liouvillian sequences* $\mathcal{L}$ is defined as the smallest D-subalgebra R of $\mathcal{S}$ that satisfies the following conditions.

1. Any order 1 operator in D has a full solution space in R.
2. If $s \in R$, then $\Sigma s \subseteq R$.
3. If $s \in R$, then the m-interlacing of s with zeros is in R for all $m \geq 1$.

Condition (3) and the closure of $\mathcal{L}$ under difference ring operations imply that $\mathcal{L}$ is closed under arbitrary interlacing.

We can now define 2-expressible sequences in a similar way to the Eulerian solutions considered in [44].

Definition 3.1.2. *Consider the family of D-subalgebras R of $\mathcal{S}$ that satisfy conditions* (2) *and* (3) *above, as well as*

4. *Any order* 2 *operator in D has a full solution space in R.*

We define $\mathcal{E}$ to be the smallest D-algebra within this family. An element of $\mathcal{E}$ is called a 2-expressible sequence. *An $\mathcal{S}$-point $s\colon M \to \mathcal{S}$ of a D-module M is* 2-expressible *if its image is contained in $\mathcal{E}$.*

Clearly a solution of an operator L is 2-expressible if and only if the corresponding $\mathcal{S}$-point of M_L is 2-expressible.

[14, Theorem 3.4] shows how Liouvillian sequences correspond to solvable Galois groups. We will give an analog for 2-expressible sequences and Eulerian Galois groups.

Definition 3.1.3. *[44, page 667] An algebraic group G is said to be* Eulerian *if there exist subgroups $1 = G_0 \subset \cdots \subset G_n = G$ such that $G_i \triangleleft G_{i+1}$ and G_{i+1}/G_i is finite or isomorphic to $\mathbb{G}_a$, $\mathbb{G}_m$, or* $\mathrm{PSL}_2(\mathbb{C})$.

Eulerian groups are closed under taking closed subgroups, quotients, and direct products [44, Lemma 2.2]. Since the connected closed subgroups of $\mathrm{SL}_2(\mathbb{C})$ are either solvable or $\mathrm{SL}_2(\mathbb{C})$ by [28, page 31] (see also [30, page 7] and [42, page 28]), the Galois group of an order 2 operator is Eulerian.

We also need to consider operators corresponding to 2-expressible sequences.

Definition 3.1.4. *We define $\mathcal{F}$ to be the smallest subset of D that satisfies the following conditions.*

1. *Any order 2 operator in D is in $\mathcal{F}$.*
2. *If $L, L' \in \mathcal{F}$, then $L \circ (\Phi - 1)$, $\Phi^i \circ L \circ \Phi^{-i}$, $\mathrm{LCLM}(L, L')$, and $L \circledS L'$ are in $\mathcal{F}$.*
3. *If L is in $\mathcal{F}$, then every right factor of L is in $\mathcal{F}$.*
4. *If $L = L(x, \Phi)$ is in $\mathcal{F}$, then so is $L(x/m, \Phi^m)$ for any $m \geq 1$.*

The minimal number of times the operations 2–4 needed to obtain $L \in \mathcal{F}$ is called the complexity *of L (the complexity is 0 if L has order 2).*

Theorem 3.1.5. *Let $s \in \mathcal{E}$. A minimal operator of s exists, is in $\mathcal{F}$, and has an Eulerian Galois group.*

Proof. Let $\mathcal{V}_\mathcal{D} = \{s \in \mathcal{S} \mid \exists\, L \in D - \{0\},\, L(s) = 0\}$, the set of elements of $\mathcal{S}$ that satisfy a recurrence relation, and let

$$A = \{s \in \mathcal{V}_\mathcal{D} \mid \text{minimal operator of } s \text{ has Eulerian Galois group}\}$$
$$B = \{s \in \mathcal{V}_\mathcal{D} \mid \text{minimal operator of } s \text{ is in } \mathcal{F}\}.$$

Solutions of order 2 operators are clearly in B, and in A by the discussion preceding Definition 3.1.4. To prove the theorem, that A and B contain $\mathcal{E}$, it remains to show that A and B are D-algebras closed under (2) and (3).

Let $s, t \in A$ be solutions to operators with Eulerian Galois groups G, H. The proof of [14, Theorem 3.4] shows that $\phi^i(s)$, Σs, and the m-interlacing of s with zeros are solutions to operators whose Galois groups are G, a subgroup of the semidirect product of G and $\mathbb{G}_a^n$ for some n, or a finite cyclic extension of G^m. Moreover, st and $s - t$ lie in a composite Picard-Vessiot ring with Galois group $G' \leq G \times H$. By [14, Lemma A.8], both st and $s - t$ are solutions to operators with Galois groups that are quotients of G'. Eulerian groups are closed under taking closed subgroups, quotients, and

direct products [44, Lemma 2.2], so $\phi(s), \Sigma s$, the m-interlacing of s with zeros, st, and $s - t$ satisfy operators with Eulerian Galois groups and so are in A. Thus $\mathcal{E} \subseteq A$.

Let $s, t \in B$ have minimal operators L, L' in $\mathcal{F}$. Then $s + t$, st, $\phi^i(s)$, and the m-interlacing of s with zeros satisfy $\mathrm{LCLM}(L, L')$, $L \circledS L'$, $\Phi^i \circ L \circ \Phi^{-i}$, and $L(x/m, \Phi^m)$, respectively, which are all in $\mathcal{F}$. By 3, their minimal operators are also in $\mathcal{F}$. □

Definition 3.1.6. *An operator is* 2-solvable *if it has a nonzero 2-expressible solution. It is* fully 2-solvable *if it has a basis of 2-expressible solutions.*

Theorem 3.1.8 below shows that any minimal operator of $s \in \mathcal{E}$ is fully 2-solvable. First we need a lemma.

Lemma 3.1.7. *If $L = L(x, \Phi)$ is in $\mathcal{F}$ then so is $L(x + c, \Phi)$ for any $c \in C$. This operation preserves the complexity of L.*

Proof. Induct on the complexity of L, noting that $\sigma_c \colon x \mapsto x + c$ commutes with 1–3 and almost-commutes with 4, i.e., if $S(L) \coloneqq L(x/m, \Phi^m)$, then $\sigma_c(S(L)) = S(\sigma_{c/m}(L))$. □

Theorem 3.1.8. *If L is in $\mathcal{F}$ then L is fully 2-solvable. In particular, any minimal operator of a 2-expressible sequence is fully 2-solvable.*

Proof. The first claim and Theorem 3.1.5 imply the second claim. To prove the first claim, induct on the complexity of L. The base case is clear. For the inductive step, suppose that $L, L' \in \mathcal{F}$ satisfy $V_{\mathcal{S}}(L), V_{\mathcal{S}}(L') \subseteq \mathcal{E}$. By Lemma 3.1.7, $L(x + i/m, \Phi) \in \mathcal{F}$. By the inductive hypothesis, $V_{\mathcal{S}}(L(x + i/m, \Phi)) \subseteq \mathcal{E}$. Let L'' be a right factor of L. Then the induction step for 2–3 is verified by the following computations.

$$
\begin{aligned}
V_{\mathcal{S}}(L \circ (\Phi - 1)) &= \bigcup_{s \in V_{\mathcal{S}}(L)} \Sigma s \quad \subseteq \mathcal{E} \\
V_{\mathcal{S}}(\Phi^i \circ L \circ \Phi^{-i}) &= \phi^i(V_{\mathcal{S}}(L)) \subseteq \mathcal{E} \\
V_{\mathcal{S}}(\mathrm{LCLM}(L, L')) &= V_{\mathcal{S}}(L) + V_{\mathcal{S}}(L') \subseteq \mathcal{E} \\
V_{\mathcal{S}}(L \circledS L') &\subseteq \mathcal{E} \\
V_{\mathcal{S}}(L'') &\subseteq V_{\mathcal{S}}(L) \subseteq \mathcal{E}
\end{aligned}
$$

Here $V_{\mathcal{S}}(L \circledS L') \subseteq \mathcal{E}$ uses equation (2.6) in Section 3.2. Finally $V_{\mathcal{S}}(L(x/m, \Phi^m))$ consists of interlacing sequences in $V_S(L(x + i/m, \Phi))$ for $0 \leq i < m$, so it is also in $\mathcal{E}$. □

A question remains: is every operator with an Eulerian Galois group 2-solvable? For more on this see Appendix F. For order 3 this follows from the proof in Section 3.5. For order 4 this follows from the fact that we give algorithms for each of the Eulerian groups listed in Lemma 3.7.6, and prove that they find solutions.

3.2 Symmetric Powers

We now state an analog for difference modules of a result by Fano, Singer, van Hoeij–van der Put [22, Proposition 3.1], and Nguyen–van der Put [37, Corollary 2.2] that determines when a D-module of dimension 3 is an almost symmetric square. Note that the difference field F needs to be a C_1-field, such as $F = \mathbb{C}(x)$ used throughout the thesis.

Proposition 3.2.1 (Symmetric Square Criterion)**.** *Let M be an irreducible D-module of dimension* 3*. Then* $\mathrm{Sym}^2(M)$ *has a D-submodule of dimension* 1 *if and only if* $M \cong \mathrm{Sym}^2(N) \otimes P$ *for some D-modules N and P of dimensions* 2 *and* 1*, respectively.*

Proof. $\Leftarrow$: The canonical D-module epimorphism $\mathrm{Sym}^2(\mathrm{Sym}^2(N)) \twoheadrightarrow \mathrm{Sym}^4(N)$ has a kernel of dimension 1. Hence $\mathrm{Sym}^2(M) \cong \mathrm{Sym}^2(\mathrm{Sym}^2(N)) \otimes \mathrm{Sym}^2(P)$ has a submodule of dimension 1 as well.

$\Rightarrow$: Let $q \in \mathrm{Sym}^2(M)$ span a D-submodule of dimension 1, i.e. $\Phi(q) = rq$ for some $r \in \mathbb{C}(x)^*$. We will show that q is irreducible (as quadratic polynomial in an $\mathbb{C}(x)$-basis x_1, x_2, x_3 of M) which is equivalent to saying that the associated bilinear form is nondegenerate. One way to see that q is irreducible is because otherwise, factors would generate a nontrivial submodule of M, but M is irreducible.

(Alternatively, identify q with its associated symmetric bilinear form on M^*. Recall that $M^* \coloneqq \mathrm{Hom}_{\mathbb{C}(x)}(M, \mathbb{C}(x))$ is a D-module under the conjugation action $\Phi(l) \coloneqq \Phi_{\mathbb{C}(x)} \circ l \circ \Phi_M^{-1}$ and enjoys properties analogous to those of its differential counterpart in [42, page 45]. The computation

$$0 = \Phi(q(x, \Phi^{-1}(y))) = \Phi(q)(\Phi(x), y) = r \cdot q(\Phi(x), y) \quad \text{for all } y \in M^*$$

verifies that $Q \coloneqq \{x \in M^* \mid q(x, M^*) = 0\}$ is a D-submodule of M^*. Also q is nonzero so Q cannot be all of M^*. Since M^* is irreducible, $Q = 0$. Therefore q is nondegenerate.)

This implies that the quadratic linear form can be reduced to $x_2^2 - x_1x_3$. This is well known, but we also give a simple algorithm in [24] that, through a sequence of elementary linear transformations, rewrites q to an $\mathbb{C}(x)$-multiple of $x_2^2 - x_1x_3$ in some $\mathbb{C}(x)$-basis x_1, x_2, x_3 of M. Alternatively, Witt's decomposition theorem [31, page 12] makes q an $\mathbb{C}(x)$-multiple of the sum of an anisotropic form x_2^2 and a hyperbolic form $-x_1x_3$.

Now that we reduced q to $x_2^2 - x_1x_3$, we will use this form to give explicit equations constraining the structure constants of the difference action on M. Write $\Phi(x_i) = \sum_j g_{ij}x_j$ with $(g_{ij}) \in \mathrm{GL}_3(\mathbb{C}(x))$. Since q spans a 1-dimensional submodule, we have $\Phi(x_2^2 - x_1x_3) = r(x_2^2 - x_1x_3)$ for some $r \in \mathbb{C}(x)^*$. Matching x_ix_j-coefficients gives the following relations.

$$\begin{aligned}
[x_1^2]: &\quad 0 = g_{21}^2 - g_{11}g_{31} \\
[x_3^2]: &\quad 0 = g_{23}^2 - g_{13}g_{33} \\
[x_1x_2]: &\quad 0 = 2g_{21}g_{22} - g_{11}g_{32} - g_{31}g_{12} \\
[x_2x_3]: &\quad 0 = 2g_{22}g_{23} - g_{12}g_{33} - g_{32}g_{13} \\
[x_2^2] + [x_1x_3]: &\quad 0 = (g_{22}^2 - g_{12}g_{32}) + (2g_{21}g_{23} - g_{11}g_{33} - g_{31}g_{13})
\end{aligned} \tag{3.1}$$

Here, the sum $[x_2^2] + [x_1x_3]$ is used to eliminate r.

We now decompose M by formally defining vector spaces N and P and giving them difference structures so that $\mathrm{Sym}^2(N) \otimes P \to M$ is a D-module isomorphism. Let N and P be $\mathbb{C}(x)$-vector spaces with respective bases $\{n_1, n_2\}$ and $\{p\}$. The $\mathbb{C}(x)$-linear map $\mathrm{Sym}^2(N) \otimes P \to M$ taking $n_in_j \otimes p$ to m_{ij} will be a D-module isomorphism if we can find $a_{ij}, b \in \mathbb{C}(x)$ defining actions $\Phi(n_i) = \sum a_{ij}n_j$ on N and $\Phi(p) = b \cdot p$ on P that satisfy

$$\begin{bmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{bmatrix} = b \begin{bmatrix} a_{11}^2 & 2a_{11}a_{12} & a_{12}^2 \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{12}a_{22} \\ a_{21}^2 & 2a_{21}a_{22} & a_{22}^2 \end{bmatrix}. \tag{3.2}$$

We now show that the matrix (g_{ij}) is diagonal, skew-diagonal, or is not of the forms A, B, C, D, where

$$A = \begin{bmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & 0 \end{bmatrix},\ B = \begin{bmatrix} 0 & * & * \\ * & * & 0 \\ 0 & * & * \end{bmatrix},\ C = \begin{bmatrix} * & * & 0 \\ 0 & * & * \\ * & * & 0 \end{bmatrix},\ D = \begin{bmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{bmatrix}.$$

To see this, suppose (g_{ij}) is of the form A, B, C, or D. Equations $[x_1^2]$ and $[x_3^2]$ of (3.1) now force A, B, and C to have a zero column, contradicting the fact that (g_{ij}) is invertible. Both (3.1) and $\det(g_{ij}) \neq 0$ force D to be diagonal or skew-diagonal (more explanation in Appendix A). Therefore,

(g_{ij}) is diagonal or skew-diagonal, or else one of $g_{11}g_{21} \neq 0$, $g_{21}g_{31} \neq 0$, $g_{13}g_{23} \neq 0$, or $g_{23}g_{33} \neq 0$ holds.

Since vertical and horizontal reflections of the square matrices (a_{ij}) and (g_{ij}) preserve the form of (3.1) and (3.2), we may further assume (g_{ij}) is diagonal or $g_{11}g_{21} \neq 0$. In the former case, set $b = g_{11}$, $a_{11} = 1$, $a_{22} = g_{22}/g_{11}$, and $a_{12} = a_{21} = 0$. In the latter case, set $b = g_{11}$, $a_{11} = 1$, $a_{12} = g_{12}/2g_{11}$, $a_{21} = g_{21}/g_{11}$, and $a_{22} = g_{32}/2g_{21}$; for this choice of a_{ij}, b, the relations (3.2) follow from (3.1) by a Gröbner basis computation in Appendix B. □

3.3 Restriction and Induction

We now define representation-theoretic operations on difference modules. The following is an isomorphism from D to $D_m := \mathbb{C}(x)[\Phi^m, \Phi^{-m}]$.

$$\begin{aligned} \psi_m : D &\longrightarrow D_m \\ \Phi &\mapsto \Phi^m \\ x &\mapsto \frac{x}{m} \end{aligned} \tag{3.3}$$

Definition 3.3.1. *Let s, t be positive integers with s dividing t. Consider the noncommutative subrings $D_s = \mathbb{C}(x)[\Phi^s, \Phi^{-s}]$ and $D_t = \mathbb{C}(x)[\Phi^t, \Phi^{-t}]$ of D.*

1. *The* restriction *of a D_s-module M to D_t is defined as the D_t-module M and denoted by $M{\downarrow_t^s}$.*
2. *The* induction *of a D_t-module N to D_s is defined as the D_s-module $D_s \otimes_{D_t} N$ and denoted by $N{\uparrow_t^s}$.*
3. *The* restriction $L{\downarrow_t^1}$ *of an operator L is a minimal operator of $\overline{1} \in M{\downarrow_t^1}$, where $M = D/DL$. $L{\downarrow_t^1}$ generates the left ideal $D_t \cap DL \subseteq D_t$. The t^{th}* section operator *is defined as $L^{(t)} := \psi_t^{-1}(L{\downarrow_t^1}) \in D$.*

Note that $L{\downarrow_m^1}$ is the same as $P(\Phi^m)$ in [14, Lemma 5.3] while $L^{(m)}$ is $P_0(\Phi)$ in [14, Lemma 5.3]. If $\overline{1}$ generates a proper D_m-submodule of $(D/DL){\downarrow_m^1}$ then we say that the order of $L^{(m)}$ is *lower than expected* (lower than the order of L).

Remark 3.3.2 (Basic properties)**.** *Let $r|s$ and $s|t$. It follows from definition that $(M{\downarrow_s^r}){\downarrow_t^s} = M{\downarrow_t^r}$ and that*

$$(N{\uparrow_t^s}){\uparrow_s^r} = D_r \otimes_{D_s} (D_s \otimes_{D_t} N) = D_r \otimes_{D_t} N = N{\uparrow_t^r}. \tag{3.4}$$

Since $\{1, \Phi^r, \Phi^{2r}, \ldots, \Phi^{s-r}\}$ *is a basis of* D_r *as a left* D_s*-module and since* $\Phi^{ir} \otimes_{D_s} N$ *is* D_s*-stable within* $D_r \otimes_{D_s} N$*, we additionally have*

$$N\uparrow_r^s\downarrow_s^r \cong (1 \otimes N) \oplus (\Phi^r \otimes N) \oplus (\Phi^{2r} \otimes N) \oplus \cdots \oplus (\Phi^{s-r} \otimes N). \tag{3.5}$$

Combining (3.4) *and* (3.5)*, we further get*

$$N\uparrow_t^r\downarrow_s^r = N\uparrow_t^s\uparrow_s^r\downarrow_s^r = \bigoplus_{i=0}^{s/r-1} (\Phi^{ir} \otimes N\uparrow_t^s).$$

For a generalization where r *need not divide* s*, see Theorem C.0.1.*

Remark 3.3.3. *It suffices to focus on* $M\downarrow_t^1$ *and* $N\uparrow_t^1$ *because results are easy to generalize to* $M\downarrow_t^s$ *or* $N\uparrow_t^s$*. After all, to generalize a result from* $D_1 = \mathbb{C}(x)[\Phi, \Phi^{-1}]$ *to* $D_s = \mathbb{C}(x)[\Phi^s, \Phi^{-s}]$*, all we have to do is replace an automorphism* ϕ *on* $\mathbb{C}(x)$ *with another automorphism* ϕ^s*.*

Restriction and induction have natural interpretations in terms of difference operators and matrix difference equations.

Let M be a D-module. If $\phi(Y) = AY$ is the associated matrix equation of M in a given basis $\mathcal{B}$, then $\phi^t(Y) = A_tY$ is the associated matrix equation of $M\downarrow_t^1$ in the same basis $\mathcal{B}$, where

$$A_t \coloneqq \phi^{t-1}(A) \cdots \phi^2(A)\phi(A)A.$$

If m is a cyclic vector of M, then $m \in M\downarrow_t^1$ need not be a cyclic vector, and so the operator $L\downarrow_t^1$ only corresponds to a D_t-submodule of $M\downarrow_t^1$.

Now let N be a D_t-module. If $\phi^t(Y) = BY$ is the associated matrix equation of N in basis $\mathcal{C}$, then $\bigcup_{i=0}^{t-1} \Phi^i \otimes \mathcal{C}$ is a basis of $N\uparrow_t^1$. In this basis, the associated matrix equation is

$$\phi(Y) = \tilde{B}Y, \qquad \tilde{B} \coloneqq \begin{bmatrix} 0 & I_{(t-1)d} \\ B & 0 \end{bmatrix},$$

where $d = \dim_F N$.

A cyclic vector n of N is also a cyclic vector of $N\uparrow_t^1$. So if R is the minimal operator of $n \in N$ (so in particular $R \in D_t$), then R is also the minimal operator of $n \in N\uparrow_t^1$, except now we view R as an element of D.

The table below summarizes the relationship we just discussed about difference modules, matrix difference equations, and difference operators.

Module	Matrix Equation	Operator
M	$\phi(Y) = AY$	$L \in D$
$M\downarrow_t^1$	$\phi^t(Y) = A_t Y$	$L\downarrow_t^1 \in D_t$ if $m \in M\downarrow_t^1$ is cyclic
N	$\phi^t(Y) = BY$	$R \in D_t$
$N\uparrow_t^1$	$\phi(Y) = \tilde{B}Y$	$R \in D$

Proposition 3.3.5 translates some of these facts into the language of difference modules for later use. But first, we make an important definition.

Definition 3.3.4. *A D_s-module M is* absolutely irreducible *if $M\downarrow_t^s$ is irreducible for all multiples t of s. An operator L is* absolutely irreducible *if $M = D/DL$ is absolutely irreducible.*

For difference and differential operators, L is absolutely irreducible when its solution space is an irreducible G°-module, where G° is the connected component of the identity of the Galois group. In the differential case this occurs when L is irreducible over algebraic extensions of $\mathbb{C}(x)$. In the difference case, such extensions are replaced with $L\downarrow_t^1$ (See Appendix D).

Proposition 3.3.5. *Let M be a D-module.*

1. *If $\dim_{\mathbb{C}(x)} M > 1$ and $\mathrm{Gal}(M)$ is solvable, then $M\downarrow_t^1$ is reducible for some $t \geq 1$ (i.e., M is not absolutely irreducible).*
2. *For some $t \geq 1$, the total Picard-Vessiot ring of $M\downarrow_t^1$ is a difference field (K, ϕ^t) and $\mathrm{Gal}(M\downarrow_t^1) = \mathrm{Gal}(M)^\circ$.*

Proof. (1): We are done if M is reducible. Suppose M is irreducible with an associated operator L. Since $\mathrm{Gal}(L)$ is solvable, some nonzero solution of L is a Liouvillian sequence [14, Theorem 3.4]. By [14, Theorem 5.1], some nonzero solution of L is an interlacing of hypergeometric sequences. By [14, Corollary 4.3], L is gauge equivalent to some $\Phi^t + f$ for some $f \in \mathbb{C}(x)$. Thus $D_t/D_t(\Phi^t + f)$ is a 1-dimensional quotient of $(D/DL)\downarrow_t^1 \cong M\downarrow_t^1$. Because $\dim_{\mathbb{C}(x)} M > 1$, we have just found a nontrivial quotient of $M\downarrow_t^1$, making $M\downarrow_t^1$ reducible.

(2): The Picard-Vessiot ring S of M has the form $\bigoplus_{i=0}^{t-1} \phi^i(R)$ for some difference ring (R, ϕ^t) where R is an integral domain. By [41, Lemma 1.26], $M\downarrow_t^1$ has the Picard-Vessiot ring (R, ϕ^t) so we set $K = \mathrm{Frac}(R)$.

By [41, Proposition 1.20], S (resp. R) is isomorphic to the coordinate ring of the algebraic group representing $\mathrm{Gal}(M)$ (resp. $\mathrm{Gal}(M{\downarrow_t^1})$) after base change to $\mathbb{C}(x)$. Since R is a connected component of S, we have $\mathrm{Gal}(M{\downarrow_t^1}) = \mathrm{Gal}(M)^\circ$. □

3.3.1 Examples of Absolute (Ir)reducibility

Let $M = D/DL$. Sections 3.5, 3.6 and 3.7 study when a *difference* equation is 2-solvable (papers [44, 16, 38] do this for differential equations). For order 4, one of the cases to check is if $M \cong M_1 \otimes M_2$ for some M_1 and M_2 with $\dim(M_1) = \dim(M_2) = 2$. Let us call this the *product-case*. One can detect the product-case by factoring the exterior square: check if $\bigwedge^2(M) \cong N_1 \oplus N_2$ for some N_1 and N_2 of dimension 3 and then write N_1 and N_2 as symmetric squares of modules of dimension 2. (The algorithm will be given in Section 6.1)

To develop such results for the difference case, there is a subtlety that needs to be studied, which we illustrate with an example.

Example 3.3.6. *In the OEIS database, the sequence A227845 has an order* 4 *minimal operator*

$$L = (x+4)^2\Phi^4 - 2(3x^2+21x+37)\Phi^3 + 2(3x^2+15x+19)\Phi - (x+2)^2.$$

Now M and $\bigwedge^2(M)$ are irreducible, but $\bigwedge^2(M{\downarrow_2^1})$ is reducible. So this example falls in the product-case, but only after restriction (Definition 3.3.1). We managed to find the following product:

$$A227845(n) = U(n-1)U(n) \tag{3.6}$$

where $U(-1) = U(0) = 1, U(1) = 2, U(2) = 7/2$, and

$$n^2U(n) = 2(3n^2-3n+1)U(n-2) - (n-1)^2U(n-4) \tag{3.7}$$

Does (3.6) *write A227845 as a product of solutions of lower order recurrences? The answer is: it depends.*

Strictly speaking, (3.7) *is a recurrence for $U(n)$ of order 4. However, it is also a recurrence for $v_0(n) := U(2n)$ of order 2. Likewise for $v_1(n) := U(2n-1)$. So if we* restrict *A227845 to even-numbered terms then equation* (3.6) *indeed writes A227845$(2n)$ as a product $v_1(n)v_0(n)$ whose factors satisfy second order recurrences. And likewise for A227845$(2n+1) = v_0(n)v_1(n+1)$. So*

A227845 is an interlacing of two sequences, each of which is a product of solutions of second order equations.

In summary: A227845 is not a symmetric product of lower-order sequences, but it becomes one if we restrict it to even-numbered terms, or to odd-numbered terms. Likewise $\bigwedge^2(M)$ is an irreducible D-module, but becomes reducible when restricted to a D_2-module.

Example 3.3.7. *In the OEIS database, both the sequences A105151 and A105216 satisfy an order* 4 *operator*

$$L = \Phi^4 - (x+4)\Phi^3 - (x+3)\Phi - 1.$$

Now $\bigwedge^2(M)$ is again irreducible, but $\bigwedge^2(M{\downarrow_2^1}) \cong N_1 \oplus N_2$ for some N_1 and N_2 of order 3*. We used ReduceOrder algorithm in Chapter 5 to write N_1 and N_2 as symmetric squares, reducing the orders to two. We then found solutions in terms of Bessel functions using [9]. Putting everything together we found:*

$$A105151(n) = \text{round}\left(c\,\text{BesselK}\left(\frac{n}{2}+\frac{1}{2},1\right)\text{BesselK}\left(\frac{n}{2}+1,1\right)\right) \tag{3.8}$$

where

$$c = \sqrt{\frac{2}{\pi}}\left(\cosh(1)\,\text{BesselI}(0,1) + \sinh(1)\,\text{BesselI}(1,1)\right).$$

The formula for A105216(n) is the same as (3.8) except for the value of c.

Motivation: Can one always decide if a recurrence of order 3 or 4 is 2-solvable or not?

This leads to the following questions:

1. Which cases can occur?
2. How to solve those cases?
3. How to *prove* non-2-solvability for recurrences that we do not solve?

Remark 3.3.8 (Remarks on questions (1), (2), and (3))**.**

1. *Let $N = \bigwedge^2(M)$ with M as in Example 3.3.6. Then N is irreducible and $N{\downarrow_2^1}$ is reducible. This raises the following question.*

   Suppose that N is not absolutely irreducible, in other words, there exists p for which $N{\downarrow_p^1}$ is reducible. Which values of p would need to be checked? This motivates Section 3.4.

2. *For order 3, see Chapters 4 and 5 of this thesis. For order 4 (Examples 3.3.6, 3.3.7) see Chapters 4 and 6.*

3. *This is the most difficult question. The only way to answer it in the differential case was through Galois theory. The main goal of the rest of Chapter 3 is to tackle this problem for the difference case. Section 3.4 gives theory that, as illustrated in the examples, will be needed for order 4. We complete this task for order 3 in Section 3.5. For order 4 this is done in Sections 3.6 and 3.7, with algorithms in Chapter 6.*

The three tasks are quite different in nature. If we limit ourselves to say orders 3 and 4, then task (1), to plausibly list all possible cases, is the easiest task. Task (2) is to design an algorithm for each case (Chapters 4, 5 and 6). The main goal for this chapter is to develop theory needed for task (3).

3.4 Semisimplicity and Absolute Irreducibility

We now develop some representation theory in the context of difference modules to decompose irreducible modules into absolutely irreducible modules.

As with differential modules [42, Exercise 2.38 (4)] (solved in Appendix E), the tensor product of two semisimple D-modules is again semisimple. Therefore the class of semisimple D-modules is closed under linear algebra operations like taking submodules, quotients, tensor products, symmetric powers, and direct sums. We now show that restriction and induction preserve semisimplicity as well. See [33, Chapter 17] for general facts on semisimplicity.

Proposition 3.4.1. *Let M be an irreducible D_s-module and N an irreducible D_t-module, with $s|t$. Then $M{\downarrow}_t^s$ and $N{\uparrow}_t^s$ are semisimple.*

Proof. (It would be sufficient to only consider $s = 1$ by Remark 3.3.3.) To show $M{\downarrow}_t^s$ is semisimple, let P be an irreducible D_t-submodule. Since M is irreducible, the D_s-submodule $P + \Phi^s(P) + \Phi^{2s}(P) + \cdots + \Phi^{t-s}(P)$ equals M. But this writes $M{\downarrow}_t^s$ as a sum of irreducible D_t-submodules $\Phi^{is}(P)$.

We next show $N{\uparrow}_t^s$ is semisimple. Let Q be a D_s-submodule of $N{\uparrow}_t^s$. Since $N{\uparrow}_t^s{\downarrow}_t^s = \bigoplus_i \Phi^{is} \otimes N$ is semisimple, there exists a projection $\alpha\colon N{\uparrow}_t^s{\downarrow}_t^s \twoheadrightarrow Q{\downarrow}_t^s$ of D_t-modules. The map

$$\tilde{\alpha}\colon N{\uparrow}_t^s \twoheadrightarrow N{\uparrow}_t^s\colon \quad n \mapsto \frac{s}{t}\sum_{i=0}^{\frac{t}{s}-1}\left(\Phi^{-is}\circ\alpha\circ\Phi^{is}\right)(n)$$

is a D_s-module homomorphism. Moreover $\tilde{\alpha}$ is a projection with image Q. This gives the D_s-module decomposition $Q \oplus \ker(\tilde{\alpha})$ of $N{\uparrow}_t^s$. □

The above contains a weak form of the analog of Clifford's theorem [34, Theorem 23.1]. Here is part of the strong form [34, Theorem 23.3]:

Proposition 3.4.2. *Let $M{\downarrow}_t^s = M_1^{n_1} \oplus \cdots \oplus M_r^{n_r}$ be the decomposition of an irreducible D_s-module M into non-isomorphic, nonzero D_t-modules M_i. Then Φ^s permutes the $M_i^{n_i}$ transitively. In particular, $n_1 = \cdots = n_r$ and $\dim_{\mathbb{C}(x)} M_1 = \cdots = \dim_{\mathbb{C}(x)} M_r$.*

Proof. The group $\langle\Phi^s\rangle = \{\ldots, \Phi^{-s}, 1, \Phi^s, \Phi^{2s}, \ldots\}$ permutes the $M_i^{n_i}$ by the uniqueness of the isotypic decomposition. This action is transitive because M is irreducible. Thus $\dim_{\mathbb{C}(x)} M_i^{n_i} = n_i \cdot \dim_{\mathbb{C}(x)} M_i$ are all equal. Since $\Phi^s(M_i^{n_i}) = [\Phi^s(M_i)]^{n_i}$, $\langle\Phi^s\rangle$ also permutes the isomorphism classes of the M_i transitively. So $\dim_{\mathbb{C}(x)} M_i$ are all equal. □

Here is an analog of [43, Section 7.4].

Theorem 3.4.3 (Mackey's irreducibility criterion)**.** *Let N be an irreducible D_t-module. Then $N{\uparrow}_t^1$ is irreducible if and only if $\Phi^i \otimes N$ is not isomorphic to N as D_t-modules for each $i = 1, 2, \ldots, t-1$.*

Proof. We first reformulate both sides of the claim. Since $N{\uparrow}_t^1$ is semisimple by Proposition 3.4.1, it is irreducible precisely when $\mathrm{End}_D(N{\uparrow}_t^1) = C$. And since N and $\Phi^i \otimes N$ are irreducible D_t-modules, $\Phi^i \otimes N$ is not isomorphic to N precisely when $\mathrm{Hom}_{D_t}(N, \Phi^i \otimes N) = 0$ (Schur's lemma). This reduces the theorem to showing that $\mathrm{End}_D(N{\uparrow}_t^1) = C$ if and only if $\mathrm{Hom}_{D_t}(N, \Phi^i \otimes N) = 0$ for $i = 1, 2, \ldots, t-1$.

Tensor-hom adjunction (or Frobenius reciprocity) gives

$$\mathrm{End}_D(N{\uparrow}_t^1) \cong \mathrm{Hom}_{D_t}(N, N{\uparrow}_t^1{\downarrow}_t^1) = \bigoplus_{i=0}^{t-1} \mathrm{Hom}_{D_t}(N, \Phi^i \otimes N).$$

The $i = 0$ summand on the right side is $\mathrm{Hom}_{D_t}(N, N) = C$, and thus $\mathrm{End}_D(N{\uparrow}_t^1) = C$ if and only if $\mathrm{Hom}_{D_t}(N, \Phi^i \otimes N) = 0$ for each $i = 1, 2, \ldots, n$. $\square$

The above describes (ir)reducibility of induced modules. For restricted modules, we have the following theorem.

Theorem 3.4.4. *If a D-module M is irreducible and $M{\downarrow}_t^1$ is reducible, then $M \cong N{\uparrow}_s^1$ for some divisor $s > 1$ of $\gcd(t, \dim_{\mathbb{C}(x)} M)$ and some irreducible D_s-module N.*

Proof. Pick a minimal divisor $s > 1$ of t for which $M{\downarrow}_s^1$ is reducible and let $N \subsetneq M{\downarrow}_s^1$ be an irreducible D_s-submodule. Since M is irreducible, the map

$$N{\uparrow}_s^1 \to M\colon \quad \Phi^i \otimes n \mapsto \Phi^i(n)$$

is surjective. To finish the proof, it suffices to show that $N{\uparrow}_s^1$ is irreducible, for then the above map is an isomorphism. It also follows that s divides $\dim_{\mathbb{C}(x)} M$ since $\dim_{\mathbb{C}(x)} N{\uparrow}_s^1 = s \cdot \dim_{\mathbb{C}(x)} N$.

Suppose $N{\uparrow}_s^1$ is reducible. By Theorem 3.4.3, we have $N \cong \Phi^i(N)$ as D_s-modules for some $0 < i < s$. Using $\Phi^s(N) = N$ and iterating the isomorphism $N \cong \Phi^i(N) \cong \Phi^{2i}(N) \cong \cdots$ give a D_s-module isomorphism

$$\psi_0\colon N \to \Phi^r(N)$$

for some r dividing s with $0 < r < s$. Let

$$\psi_i = \Phi^{ir} \circ \psi_0 \circ \Phi^{-ir}\colon \Phi^{ir}(N) \to \Phi^{(i+1)r}(N).$$

Then $c \coloneqq \psi_{s/r-1} \circ \ldots \circ \psi_1 \circ \psi_0$ is in $\mathrm{End}_{D_s}(N) = C$. After rescaling each ψ_i by $c^{-r/s}$, we may assume $c = 1$. We further define

$$\Psi_i = \psi_{i-1} \circ \cdots \circ \psi_0\colon N \to \Phi^{ir}(N)$$

and

$$\Psi\colon N \to M\colon \quad n \mapsto \sum_{i=0}^{s/r-1} \Psi_i(n).$$

Since $\Psi_1 = \psi_0\colon N \to \Phi^r(N)$ is an isomorphism, if $n \in N$ then $\Phi^r(n) = \Psi_1(n')$ for some $n' \in N$. From $\Psi_{s/r} = c = 1 = \Psi_0$ we get

$$\Phi^r\left(\Psi(n)\right) = \sum_{i=0}^{s/r-1} \Phi^r \Psi_i(n) = \sum_{i=0}^{s/r-1} \Psi_{i+1}(n') = \Psi(n')$$

which shows that $\Psi(N) \subsetneq M\downarrow_r^1$ is a D_r-module. Thus $M\downarrow_r^1$ is reducible, contradicting the minimality of s. □

Repeated use of Theorem 3.4.4 implies that an irreducible D-module is induced from an absolutely irreducible module.

Corollary 3.4.5. *If a D-module M is irreducible and $M\downarrow_t^1$ is reducible, then $M\downarrow_p^1$ is reducible for some prime divisor p of $\gcd(t, \dim_{\mathbb{C}(x)} M)$.*

Proof. By Theorem 3.4.4, $M \cong N\uparrow_s^1$ for some D_s-module N and some divisor $s > 1$ of $\gcd(t, \dim_{\mathbb{C}(x)} M)$. Let p be a prime divisor of s. By Remark 3.3.2, $M\downarrow_p^1 \cong N\uparrow_s^1\downarrow_p^1 \cong \bigoplus_{i=0}^{p-1} \left(\Phi^i \otimes N\uparrow_s^p\right)$ is reducible. □

3.5 Main Theorem for Order 3

We now prove our main result for order 3 by reducing to the essential case (Lemma 3.5.2) where the total Picard-Vessiot ring is a field.

Theorem 3.5.1. *Let M be a D-module of dimension 3. If a nonzero S-point of M is 2-expressible, then*

1. *M is reducible over $\mathbb{C}(x)$, or*
2. *$M \cong N\uparrow_3^1$ for some D_3-module N, or*
3. *$M \cong \mathrm{Sym}^2(N) \otimes P$ for some D-modules N and P of dimensions 2 and 1, respectively.*

We can now deduce the restatement of Theorem 3.5.1 in Chapter 1: let L be a 2-solvable operator of order 3, so that Theorem 3.5.1 applies to M_L. If M_L is reducible, so is L. If $M \cong N\uparrow_3^1$, then L is gauge equivalent to some $\Phi^3 + f$ per the discussion in Section 3.3. Finally suppose that $M_L \cong \mathrm{Sym}^2(N) \otimes P$ and L_2 and L_1 are operators associated to N and P, respectively. If $L_2^{\circledS 2}$ has order 2, then L is reducible. Otherwise $L_2^{\circledS 2}$ has order 3 and L is gauge equivalent to $L_2^{\circledS 2} \circledS L_1$.

Proof of Theorem 3.5.1. If M is reducible, (1) holds. If M is irreducible but not absolutely irreducible, then $M \cong N\uparrow_3^1$ by Theorem 3.4.4.

We may now assume M is absolutely irreducible. Since L is 2-solvable, $\mathrm{Gal}(L)$ is Eulerian by Theorem 3.1.5. By Proposition 3.3.5 (2), there exists $t \geq 1$ for which the total Picard-Vessiot ring of $M{\downarrow_t^1}$ is a difference field (K, ϕ^t) and $\mathrm{Gal}(M{\downarrow_t^1}) = \mathrm{Gal}(M)^\circ$. Thus $\mathrm{Gal}(M{\downarrow_t^1})$ is Eulerian and connected.

Let $Q_1 \oplus Q_2 \oplus \cdots \oplus Q_r$ be an irreducible decomposition of $\mathrm{Sym}^2(M)$. Then

$$\mathrm{Sym}^2(M{\downarrow_t^1}) = Q_1{\downarrow_t^1} \oplus Q_2{\downarrow_t^1} \oplus \cdots \oplus Q_r{\downarrow_t^1}.$$

By Proposition 3.4.2, $Q_i{\downarrow_t^1}$ is a direct sum of irreducible D_t-modules of the same dimension. If we now apply Lemma 3.5.2 below to the difference module $M{\downarrow_t^1}$, we must have $r = 2$, $\dim_{\mathbb{C}(x)} Q_1{\downarrow_t^1} = 1$, and $\dim_{\mathbb{C}(x)} Q_2{\downarrow_t^1} = 5$. In particular, $\dim_{\mathbb{C}(x)} Q_1 = 1$. The Symmetric Square Criterion (Proposition 3.2.1) then gives $M \cong \mathrm{Sym}^2(N) \otimes P$ for some D-modules N and P of dimensions 2 and 1, respectively. □

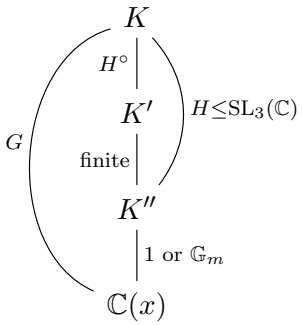
To finish the proof of Theorem 3.5.1, we need

Lemma 3.5.2. *Let M be an absolutely irreducible D-module of dimension 3 whose total Picard-Vessiot ring is a field and whose Galois group is Eulerian and connected. Then $\mathrm{Sym}^2(M)$ is the direct sum of two irreducible D-modules of dimensions 1 and 5, respectively.*

Proof. $G := \mathrm{Gal}(M/\mathbb{C}(x))$ embeds into $\mathrm{GL}_3(\mathbb{C})$ because $\dim_{\mathbb{C}(x)} M = 3$. Since G is connected, the group homomorphism

$$G \hookrightarrow \mathrm{GL}_3(\mathbb{C}) \xrightarrow{\det} \mathbb{G}_m$$

has image 1 or $\mathbb{G}_m$ and a kernel $H \leq \mathrm{SL}_3(C)$. Let K be the total Picard-Vessiot ring of M over $\mathbb{C}(x)$. We further let $K' = K^{H^\circ}$ and $K'' = K^H$. Then K/F decomposes as



First we show $K'/\mathbb{C}(x)$ is a total Picard-Vessiot extension with Galois group G/H° and G/H° is either 1 or $\mathbb{G}_m$. Since H° is characteristic in H and $H \trianglelefteq G$, we have $H^\circ \trianglelefteq G$ and $K'/\mathbb{C}(x)$ is a total Picard-Vessiot extension with Galois group G/H°. As a quotient of G, G/H° is connected. If $G/H = 1$, then G/H° is finite and connected, hence 1. If instead $G/H = \mathbb{G}_m$, then G/H° is connected of dimension 1 with quotient $\mathbb{G}_m$. The only connected algebraic groups of dimension 1 are $\mathbb{G}_a$ and $\mathbb{G}_m$ [47, Theorem 3.4.9], so $G/H^\circ = \mathbb{G}_m$.

Next, note that $H^\circ \leq H \leq SL_3(\mathbb{C})$. Since H° is a closed subgroup of the Eulerian group G, H° is again Eulerian. By [44, Proposition 4.1], the possibilities for a group like H° that is both Eulerian and in $\mathrm{SL}_3(\mathbb{C})$ are:

(a) H° is finite.

(b) $V(M)$ has an H°-submodule of dimension 2.

(c) $V(M)$ has an H°-submodule of dimension 1.

(d) $H^\circ \cong \mathrm{PSL}_2(\mathbb{C})$, and $V(M) \cong R_2$ as $\mathrm{PSL}_2(\mathbb{C})$-modules. Here R_2 is the degree 2 homogeneous part of $\mathbb{C}[x, y]$, equipped with the standard action of $\mathrm{SL}_2(\mathbb{C})$ on degree 1 and extended to all degrees. Also the $\mathrm{PSL}_2(\mathbb{C})$-action on $V(M)$ is via the isomorphism $H^\circ \cong \mathrm{PSL}_2(\mathbb{C})$.

Since M is absolutely irreducible, the contrapositive of Proposition 3.3.5(1) shows that G is not solvable. To conclude the proof, we will show that Cases (a)–(c) force H° (and hence G) to be solvable and so cannot occur. We will show that Case (d) gives our desired result.

Case (a): We must have $H^\circ = 1$ (solvable). This rules out Case (a).

Case (b): Since M is irreducible, an H°-submodule $U \subset V(M)$ of dimension 2 cannot be G-stable in light of Proposition 2.3.2. Thus there exists $g \in G$ such that $g(U) \neq U$, and $g(U)$ is an H°-submodule because $H^\circ \trianglelefteq G$. A dimension count gives $\dim_{\mathbb{C}(x)}(U \cap g(U)) = 1$, reducing to Case (c).

Case (c): Let $U \subset V(M)$ be a H°-submodule of dimension 1. The restriction of the irreducible G-representation $V(M)$ to $H^\circ \trianglelefteq G$ is a semisimple representation $V(M) = g_1(U) \oplus g_2(U) \oplus g_3(U)$ over some $g_1, g_2, g_3 \in G$. Since each $g_i(U)$ is 1-dimensional, $\mathrm{Gal}(M/K')$ is a subgroup of $\mathbb{G}_m^3$. But K is the total Picard-Vessiot field of M over $\mathbb{C}(x)$ (and hence over K' too), so

$$H^\circ = \mathrm{Gal}(K/K') = \mathrm{Gal}(M/K') \leq \mathbb{G}_m^3.$$

Therefore H° is a solvable group. This rules out Case (c).

Case (d): From $V(M) \cong R_2$, we get

$$V(\mathrm{Sym}^2(M)) \cong \mathrm{Sym}^2(V(M)) \cong \mathrm{Sym}^2(R_2).$$

The $\mathrm{PSL}_2(\mathbb{C})$-module $\mathrm{Sym}^2(R_2)$ is known to be a direct sum of irreducible modules of dimensions 1 and 5. Thus $V(\mathrm{Sym}^2(M))$ decomposes in the same way into some irreducible H°-modules, say W_1 and W_2, of dimensions 1 and 5 respectively. Since the dimensions of W_1 and W_2 are different and since these are the *only* irreducible H°-submodules of $V(\mathrm{Sym}^2(M))$, the irreducible H°-submodule $g(W_i)$ must equal W_i within $V(\mathrm{Sym}^2(M))$, for each $g \in G$ and $i = 1, 2$. In other words, the W_i are G-stable and $V(\mathrm{Sym}^2(M)) = W_1 \oplus W_2$ is a decomposition of G-modules. By Proposition 2.3.2, $\mathrm{Sym}^2(M)$ has corresponding irreducible D-submodules of dimensions 1 and 5. This produces the desired direct sum decomposition of $\mathrm{Sym}^2(M)$. □

3.6 Main Theorems for Order 4

Theorem 3.6.1. *Let L be an irreducible operator of order 4. Then L is 2-solvable if and only if* $\mathrm{Gal}(L)$ *is Eulerian.*

Proof. Theorem 3.1.5 gives "⇒". For the converse, the case where L is not absolutely irreducible will be handled in Chapter 4. The remaining cases are listed in Lemma 3.7.6. Algorithms for these cases are given in Chapter 6. Sections 3.7.1 and 3.7.2 show that if $\mathrm{Gal}(L)$ is Eulerian, then our algorithms will find a 2-expressible solution. □

Theorem 3.6.2. *If $L \in D = \mathbb{C}(x)[\Phi, \Phi^{-1}]$ is a fourth order linear difference operator that is 2-solvable, then one of the following holds*

1. *L is reducible in D;*
   (Algorithm RightFactors in Maple)
2. *L is irreducible but not absolutely irreducible;*
   (Algorithm AbsFactorization in Section 4.1)
3. *$L^{(2)}$ is gauge equivalent to $L_{2a} \circledS L_{2b}$ for some second order operators $L_{2a}, L_{2b} \in D$;*
   (Algorithm in Section 6.1)

4. L is gauge equivalent to some $L_2^{\circledS 3} \circledS (\Phi - r)$.
(Algorithm in Section 6.2)

3.7 Proof of Theorem 3.6.2

Definition 3.7.1. *[16, Definition 14] Let G be a subgroup of $\mathrm{GL}_n(\mathbb{C})$ acting irreducibly on the vector space V of dimension n over $\mathbb{C}$. Then G is called* imprimitive *if, for $k > 1$, there exist subspaces $V_1, V_2, \ldots, V_k$ such that $V = V_1 \oplus V_2 \oplus \ldots \oplus V_k$ and, for each $g \in G$, the mapping $V_i \to g(V_i)$ is a permutation of the set $S = \{V_1, V_2, \ldots, V_k\}$. An irreducible group $G \subseteq \mathrm{GL}_n(\mathbb{C})$ which is not imprimitive is called* primitive.

Remark 3.7.2. *If G is imprimitive, this implies that a finite-index subgroup leaves V_1 fixed. This means that $V(L)$ is reducible under a finite-index subgroup, and by Theorem D.0.1, we have that L is not absolutely irreducible.*

Remark 3.7.3. *Every subspace is closed under multiplying by scalars, so if $H \subseteq G \subseteq \mathbb{C}^* H \subseteq \mathrm{GL}(W)$ then a subspace of W is G-invariant iff it is H-invariant.*

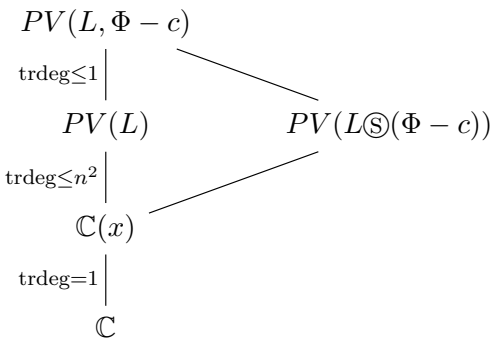
Lemma 3.7.4. *Let G be the Galois group of L and let $G_1 = G \cap \mathrm{SL}_n$. Let W be obtained from $V(L)$ by linear algebra constructions (tensor products, exterior powers, etc.). If $\mathbb{C}^* \subseteq G$ then a subspace of W is G-invariant if and only if it is G_1-invariant.*

Proof. By Remark 3.7.3 it suffices to show $G \subseteq \mathbb{C}^* G_1$. Let $g \in G$ and let d be the determinant of g acting on $V(L)$. Then $d^{-1/n}$ is in G by the hypothesis, and so $d^{-1/n} g$ is in G, but also in SL_n, and thus in G_1. It follows that $g \in \mathbb{C}^* G_1$. □

We use the above lemma to deal with a technical issue that in general, G is an algebraic subgroup of GL_n but the classification we use from [38] lists algebraic subgroups of SL_n. The lemma shows that this classification is sufficient; as long as we make sure that $\mathbb{C}^*$ is in G, intersecting G with SL_n will not change the factorization pattern (orders of the factors) of symmetric and exterior powers.

Another construction that does not change such factorization patterns is replacing L with $L \circledS L_1$ for any order 1 operator L_1. We will show that taking $L_1 = \Phi - c$ for suitable c makes Lemma 3.7.4 applicable.

Lemma 3.7.5. *Let $H \subseteq \mathbb{C}^*$ be the Galois group of $PV(L, \Phi - c)$ over $PV(L)$. There exists $c \in \mathbb{C}^*$ for which $H = \mathbb{C}^*$. Then the Galois group of $PV(L\circledS(\Phi - c))$ over $\mathbb{C}(x)$ contains $\mathbb{C}^*$.*



Proof. Let N be larger than the transcendence degree of $PV(L)$ over $\mathbb{C}$. Let $\lambda_1, \ldots, \lambda_N \in \mathbb{R}$ be $\mathbb{Q}-$linearly independent. Let $c_i = e^{\lambda_i}$. It is easy to prove that $c_1^x, \ldots, c_N^x$ are algebraically independent ([32, page 8]), so $\exists\, c \in \{c_1, \ldots, c_N\}$ such that the extension of $PV(L, \Phi - c)$ over $PV(L)$ is transcendental. That implies that H (an algebraic subgroup of $\mathbb{C}^*$) is infinite, and hence equal to $\mathbb{C}^*$. But H acts on solutions of $L\circledS(\Phi - c)$, since these are solutions of L (on which H acts trivially) times the solution c^x of $\Phi - c$, on which H acts as $\mathbb{C}^*$. □

Lemma 3.7.6. *Let L be an absolutely irreducible difference operator of order 4. Let $\tilde{L} = L\circledS(\Phi - c)$ where $c \in \mathbb{C}^*$ is chosen in such a way that the Galois group G of $\tilde{L}$ contains $\mathbb{C}^*$. Let $G_1 = G \cap \mathrm{SL}_4$ and let G_1° be the identity component of G_1. (The fact that G contains $\mathbb{C}^*$ implies that G_1° is also equal to $G^\circ \bigcap \mathrm{SL}_4$). Then G_1° is primitive (see Remark 3.7.2), G_1/G_1° is cyclic (see [14, Theorem 2.1] and [41, Proposition 1.20]), and G_1° is one of the following groups:*

Table 3.1: Decompositions of the Infinite Primitive Groups

G_1°	$\bigwedge^2 V$	$\mathrm{Sym}^2 V$	G_1° Eulerian?	2-solvable?	Algorithms
$\mathrm{So}_4(\mathbb{C})$	3^2	1,9	Yes	Yes	Section 6.1
$\mathrm{SL}_2(\mathbb{C})$	1,5	3,7	Yes	Yes	Section 6.2
$\mathrm{SL}_4(\mathbb{C})$	6	10	No, Appendix F	No, Theorem 3.1.8	-
$\mathrm{Sp}_4(\mathbb{C})$	1,5	10	No, Appendix F	No, Theorem 3.1.8	-

The table lists the dimensions (from [38, Table 4.1]) of the irreducible G_1°-submodules of $\bigwedge^2 V(\tilde{L})$ and $\mathrm{Sym}^2 V(\tilde{L})$. [38, Proposition 5] shows that these are also G_1-invariant (and hence G-invariant since $G_1 \subseteq G \subseteq \mathbb{C}^* G$) except possibly the two 3-dimensional spaces in the first row (in which case they are invariant under an index 2 subgroup of G_1). By Galois correspondence, this means that if

$M = D/D\tilde{L}$ then the modules $\bigwedge^2 M$ and $\mathrm{Sym}^2 M$ are direct sums of irreducible[1] submodules, with dimensions indicated in the table, except possibly the 3^2 in the first row, as these factors may only arise after replacing M with $M{\downarrow_2^1}$. The same is then also true for D/DL instead of $D/D\tilde{L}$ because replacing L with $\tilde{L} = L \circledS (\Phi - c)$ does not change any factorization pattern, see Remark 3.7.3.

We omitted rows from [38, Table 4.1] where G_1/G_1° is not cyclic because such groups do not appear in the difference case. We also deleted the last three columns from [38, Table 4.1] since those are not needed here.

In Theorem 3.6.2, cases 1 and 2 cover the cases when L is not absolutely irreducible. If L is absolutely irreducible, and if G_1° is $\mathrm{So}_4(\mathbb{C})$, then section 3.7.1 will show that case 3 in Theorem 3.6.2 holds, and if G_1° is $\mathrm{SL}_2(\mathbb{C})$ then section 3.7.2 will show that case 4 holds. This completes the proof of Theorem 3.6.2.

3.7.1 The Case When $G_1^\circ = \mathrm{SO}_4(\mathbb{C})$

Definition 3.7.7. *If $g \in \mathrm{GL}_2(\mathbb{C})$ then g acts on $\mathbb{C}^2$. Let $\rho_3(g) \in \mathrm{GL}_3(\mathbb{C})$ denote the corresponding action on $\mathbb{C}^3$, identified with $\mathrm{Sym}^2(\mathbb{C}^2)$, and let $\rho_4(g) \in \mathrm{GL}_4(\mathbb{C})$ denote the corresponding action on $\mathbb{C}^4$, identified with $\mathrm{Sym}^3(\mathbb{C}^2)$.*

The maps $\rho_3 : \mathrm{GL}_2 \to \mathrm{GL}_3$ and $\rho_4 : \mathrm{GL}_2 \to \mathrm{GL}_4$ are group homomorphisms. If G is Galois group in case 3 in Theorem 3.5.1 then $\mathbb{C}^*G = \rho_3(\mathrm{GL}_2)$. Similarly, in the second row in Table 3.1 we have $\mathbb{C}^*G = \rho_4(\mathrm{GL}_2(\mathbb{C}))$. (Due to Lemma 3.7.5 we may assume wlog that $\mathbb{C}^* \subseteq G$.) If $g = (a_{ij}) \in \mathrm{GL}_2(\mathbb{C})$, then $\rho(g)$ is the 3×3 matrix in the right-hand side of Equation (3.2). It is easy to check that the kernel of ρ_3 is $\{\pm I\}$.

Definition 3.7.8. *We say that a group G* acts projectively *on a $\mathbb{C}$-vector space V when there is a natural action of G on the projective space $P(V) = (V - \{0\})/\mathbb{C}^*$.*

Proposition 3.7.9. *Let L, $\tilde{L}$ and G be as in Lemma 3.7.6 and suppose that we are in row 1. Let $L_6 = \bigwedge^2(L)$ so $\bigwedge^2 M \cong D/DL_6$ where $M = D/DL$.*

1. *L_6 or $L_6{\downarrow_2^1}$ has two absolutely irreducible factors, say L_{3a} and L_{3b}, of order 3.*
2. *L_{3a} and L_{3b} are 2-solvable.*

[1] absolutely irreducible, because they are not just G-irreducible but also G°-irreducible

3. L or $L{\downarrow_2^1}$ is projectively equivalent to a symmetric product.

Proof. 1. As explained after the table, $\bigwedge^2 M$ or $\bigwedge^2 M{\downarrow_2^1}$ is a direct sum of two absolutely irreducible modules of dimension 3. This means that L_6 or $L_6{\downarrow_2^1}$ is an LCLM of two absolutely irreducible factors of order 3, say L_{3a} and L_{3b}. In the remainder of the proof we assume the former (the proof for the latter case is the same, just replace L with $L{\downarrow_2^1}$).

2. All solutions of L_6, and hence of L_{3a} and L_{3b}, are in the Picard-Vessiot extension of L. But G is an Eulerian group, and by the proof of Theorem 3.5.1 the operators L_{3a} and L_{3b} are 2-solvable, specifically, they are gauge equivalent to $L_{2a}^{\circledS 2} \circledS (\Phi - r_a)$ and $L_{2b}^{\circledS 2} \circledS (\Phi - r_b)$ for some operators L_{2a} and L_{2b} and rational functions r_a, r_b.

3. Let K_L be the Picard-Vessiot extension of L, and let K be the Picard-Vessiot extension of the LCLM of L, L_{2a}, L_{2b}, $\Phi - r_a$, $\Phi - r_b$. Let $\tilde{G}$ be the difference Galois group of K. By the Galois correspondence, G, the difference Galois group of K_L, is a quotient group of $\tilde{G}$. Specifically, $G = \tilde{G}/N$ where N is the normal subgroup of $\tilde{G}$ that acts trivially on K_L. So we can write any $g \in G$ as $\tilde{g}N$ for some $\tilde{g} \in \tilde{G}$.

Solutions of L, and hence solutions of L_6, are elements of K_L. Solutions of L_{3a} are also in K_L since L_{3a} is a right-factor of L_6. This implies that solutions of $L_{2a}^{\circledS 2} \circledS (\Phi - r_a)$ are also elements of K_L, and hence, G acts on these solutions. If b_1, b_2, b_3 is a basis of solutions of $L_{2a}^{\circledS 2} \circledS (\Phi - r_a)$, then the action of G on $\mathrm{SPAN}_{\mathbb{C}}\{b_1, b_2, b_3\}$ is as in case (d) in Lemma 3.5.2. However, this does not imply that G acts on $V(L_{2a})$ because that is not necessarily a subset of K_L. This is why the extension $K_L \subseteq K$ is needed.

Although G need not act on $V(L_{2a})$ we will show that G does act projectively on it (Definition 3.7.8). If $g \in G$ then we can write $g = \tilde{g}N$ for some $\tilde{g} \in \tilde{G}$. This $\tilde{g}$ does act on $V(L_{2a})$ since it is a subset of K. It remains to show that N can only act by scalar multiplication on $V(L_{2a})$. This follows from the fact that G can only act as scalar multiplication on the 1-dimensional solution space of $\Phi - r_a$ and the fact that $\rho_3 := \mathrm{GL}_2 \to \mathrm{GL}_3$ has kernel $\{\pm 1\} \subset \mathbb{C}^*$.

Hence G acts projectively on $V(L_{2a})$, meaning, it acts on $P(V(L_{2a}))$. Applying the same to L_{2b} we see that G also acts projectively on $V(L_{2a}) \otimes V(L_{2b})$. A computation shows that the corresponding

projective action on $\bigwedge^2(V(L_{2a}) \otimes V(L_{2b}))$ coincides with that on $V(L_{3a}) \oplus V(L_{3b}) = V(L_6) \cong \bigwedge^2(V(L))$.

If V_1, V_2 are 4-dimensional G-modules, and if the projective action of G on $\bigwedge^2 V_1$ and $\bigwedge^2 V_2$ is the same, then the same is true for the projective action on V_1 and V_2. This is because the kernel of the map $\mathrm{GL}_4 \to \mathrm{GL}_6$ that translates an action on $\mathbb{C}^4$ to an action on $\bigwedge^2 \mathbb{C}^2$ also has kernel $\{\pm 1\}$.

Apply this to $V_1 = V(L_{2a}) \otimes V(L_{2b}) \cong V(L_{2a} \circledS L_{2b})$ and $V_2 = V(L)$, then Proposition 2.3.7 shows that $L_{2a} \circledS L_{2b}$ and L are projectively equivalent. □

3.7.2 The Case When $G_1^\circ = \mathrm{SL}_2(\mathbb{C})$

Proposition 3.7.10. *Let L be as in Lemma 3.7.6 and suppose that we are in row 2. Let $M = D/DL$ and let $M_{10} = \mathrm{Sym}^2(M)$ (the subscript indicates its dimension).*

1. *$M_{10} = M_3 \oplus M_7$ for some absolutely irreducible sub-modules M_3 and M_7 of dimensions 3 and 7 respectively.*
2. *$M_3 \cong \mathrm{Sym}^2(N) \otimes P$ for some D-modules N and P of dimensions 2 and 1, respectively.*
3. *M is projectively equivalent to $\mathrm{Sym}^3(N)$, i.e. $M \cong \mathrm{Sym}^3(N) \otimes Q$ for some 1-dimensional D-module Q.*

Proof. 1. We have $\mathbb{C}^*G = \rho_4(\mathrm{GL}_2(\mathbb{C}))$, so any g in G can be written as $\rho_4(A)$ for some 2×2 matrix $A = (a_{ij})$. Let $\rho : \mathrm{GL}_4 \to \mathrm{GL}_{10}$ translate an action on $\mathbb{C}^4$ to an action on $\mathbb{C}^{10}$, identified with $\mathrm{Sym}^2(\mathbb{C}^4)$. It is not difficult to choose a basis (for more details, see [24]) in such a way that $\rho(\rho_4(A))$ is a block diagonal matrix, with blocks of size 3×3 and 7×7, explicitly proving that the G-module $\mathrm{Sym}^2(V)$ in row 2 in the table is a direct sum of irreducible submodules of dimensions 3 and 7, that are still irreducible as G_1°-modules. By Galois correspondence, M must therefore be a direct sum of absolutely irreducible submodules of dimensions 3 and 7, say M_3 and M_7.

2. Since G is Eulerian, and M_3 is absolutely irreducible, item 2 now follows from Theorem 3.5.1.

3. Now as in the proof of Proposition 3.7.9, G need not act on the solution of N. For this, we need to switch to a larger PV extension. But G does act projectively on this space. One can verify (e.g. by computing every mentioned group action as an explicit 4×4 matrix, 10×10 matrix, and 3×3

matrix, 2×2 matrix, and again 4×4 matrix, all written in terms of a_{ij}, $i, j \in \{1, 2\}$) that G has the same projective action on the solutions of $\mathrm{Sym}^3(N)$ as on the solutions of M, and thus these modules are projectively equivalent, similar to the proof of Proposition 3.7.9.

□

CHAPTER 4

ABSOLUTE FACTORIZATION

This chapter is reproduced from joint work [6] and [8].

In this section, we examine when, and in what sense, an irreducible operator can be "factored" further, or else be "absolutely irreducible". The motivation is that this is needed when we want to extend our work to order > 3.

A submodule of $M{\downarrow}_m^1$ is simply a subset of M that is also a D_m-module. Since every D-module is a D_m-module, but not vice versa, $M{\downarrow}_m^1$ could have more submodules than M.

Definition 4.0.1. *Let m be in $\mathbb{Z}^+$ and let L be in D_m. We define* $\operatorname{order}_m(L) \coloneqq \operatorname{order}(L)/m$, *the highest power of Φ^m that appears in L. Note that* $\operatorname{order}_m(L) = \operatorname{order}(\psi_m^{-1}(L))$ *where $\psi_m : D \to D_m$ was defined in Equation (3.3).*

Theorem 4.0.2. *Let L be irreducible in D and let $M = D/DL$. Let m be in $\mathbb{Z}^+$. Then the following are equivalent.*

1. *$M{\downarrow}_m^1$ is a reducible D_m-module.*
2. *$L{\downarrow}_m^1$ is reducible in D_m or* $\operatorname{order}_m(L{\downarrow}_m^1) < \operatorname{order}(L)$.
3. *$L^{(m)}$ is reducible in D or* $\operatorname{order}(L^{(m)}) < \operatorname{order}(L)$.
4. *$M{\downarrow}_p^1$ is reducible for some prime p dividing* $\gcd(m, \dim(M))$.

Proof.

(1) $\Leftrightarrow$ (2): If $\bar{1} = 1 + DL$ generates $M{\downarrow}_m^1$, then its minimal operator $L{\downarrow}_m^1$ has the expected order, and is reducible in D_m if and only if $M{\downarrow}_m^1$ is reducible. If $\bar{1}$ does not generate $M{\downarrow}_m^1$, then it generates a nontrivial submodule, and $L{\downarrow}_m^1$ has lower order than expected.

(2) $\Leftrightarrow$ (3): Apply the isomorphism $\psi_m : D \to D_m$ (Equation (3.3)).

(4) $\Rightarrow$ (1): Clear because $D_m \subseteq D_p$.

(1) $\Rightarrow$ (4): This is precisely Corollary 3.4.5 . $\square$

As mentioned in the above proof, if $M = D/DL$ then $\bar{1} = 1 + DL$ generates M as a D-module, but it might not generate M as D_m-module, in other words, $D_m \cdot \bar{1} \subsetneq M{\downarrow}_m^1$ is a proper submodule. That case corresponds to Step 1b in the algorithm below. The other option is that $\bar{1}$ does generate $M{\downarrow}_m^1$. In that case, having a proper submodule is equivalent to $L{\downarrow}_m^1$ having a proper factor in D_m, and $L^{(m)}$ having a proper factor in D, which corresponds to Step 1c in the algorithm below.

4.1 First Absolute Factorization Algorithm

Theorem 4.0.2 (4) immediately gives the correctness of the following algorithm. The implementation is in [24].

Algorithm: `AbsFactorization`

Input: An irreducible operator $L \in D$.[1]

Output: `Absolutely Irreducible` if L is absolutely irreducible; otherwise $[p, \{R_i\}]$ where p is a prime and where each R_i right-divides $L^{(p)}$ and $\psi_p(R_i)$ generates a nontrivial submodule of $M{\downarrow}_p^1$, with $M = D/DL$.

1. For each prime factor p of order(L) :
   (a) Compute $L^{(p)}$ from Definition 3.3.1.
   (b) If order$(L^{(p)}) <$ order(L), then return $[p, \{1\}]$.
   (c) Compute $S \coloneqq \text{RightFactors}\left(L^{(p)}, \frac{\text{order}(L)}{p}\right)$.
   (d) If $S \neq \varnothing$, then return $[p, S]$ and stop.
2. Return `Absolutely Irreducible`.

The main step is to factor $L^{(p)}$ in D. Section 4.2 below discusses factoring and gives an example to illustrate an efficiency issue, which will be addressed in Section 4.3.

[1] Factoring is explained in [4, 50]. The Maple command `RightFactors`, implemented by van Hoeij, factors L if it is reducible. By default, it factors in $C(x)[\Phi]$ where C is the smallest field of constants over which the input is defined. To ensure a full factorization over $\mathbb{C}$ one may need to add field extensions to an optional input of `RightFactors` to ensure that C is large enough.

4.2 Factorization in D

Manuel Bronstein showed that finding d'th order right-factors of L can be done by computing hypergeometric solutions of a system $\phi(Y) = AY$ where $A = \bigwedge^d(C_L)$. He also made significant progress towards solving such systems, see [4] for more details.

If we know the so-called *types* of hypergeometric solutions for a system $\phi(Y) = AY$, then finding hypergeometric solutions reduces to computing *polynomial solutions* of related systems $\phi(P) = \lambda_i^{-1}AP$, see [4, Sections 4 and 5] for details. Thus, a key goal in [4] is to construct a set of *candidate types* (denoted $\mathcal{H}_2 \subset \mathbb{C}(x)$ in [4]). This set should be as small as possible, because for each $\lambda_i \in \mathcal{H}_2$ we have to compute[2] the polynomial solutions of $\phi(P) = \lambda_i^{-1}AP$.

We will call S a set of *candidate determinants* for L if $\det(R)$ provably appears in S up to $\sim$ for every order-d right-factor $R \in D$ of L. Our goal is to minimize S because it equals the set of candidate types for $\phi(Y) = AY$ when $A = \bigwedge^d(C_L)$.

Example 4.2.1. *The cost of Step 1(c) in algorithm* `AbsFactorization` *depends on the number of candidates in the factorization, which in turn depends on the number of factors of the leading and trailing coefficients. To illustrate this we constructed an example of a fourth order difference operator where $L^{(2)}$ is reducible and has many small factors in the leading and trailing coefficients: $L = a_4\Phi^4 + a_3\Phi^3 + \cdots + a_0\Phi^0$, where* $a_4 = (x+4)^2(x+5)^2(2x+7)^2(2x+9)^2(4x-11)(524160x^8 + 9391200x^7 - 118179432x^6 - 253541284x^5 - 339259113x^4 - 283416626x^3 - 140532705x^2 - 35130024x - 2220048)$, $a_3 = 16(x+4)^2(2x+7)^2(524160x^{12} + 15113280x^{11} - 364158816x^{10} - 4278491572x^9 - 9186978746x^8 + 12166953346x^7 - 86741410290x^6 - 843333775440x^5 - 2144077451746x^4 - 3001904754612x^3 - 2506144851117x^2 - 1178353117620x - 242095406175)$, $a_2 = (-137438945280x^{17} - 6482503372800x^{16} - 90355220358144x^{15} - 154953056569984x^{14} + 8139627355615616x^{13} + 69179680108818000x^{12} + 277321791062784832x^{11} + 698868352509149328x^{10} + 1236863662787672992x^9 + 1625448731323698944x^8 + 1626145247262854144x^7 + 1235819925815197696x^6 + 686291085150978048x^5 + 244593652122419200x^4 + 2404529004281856 0x^3 - 27607241721839616x^2 - 15602879836717056x - 2930851407200256)$, $a_1 = 32768(x+2)^2(2x+3)^2(1048320x^{12} + 38613120x^{11} - 475672512x^{10} - 11499544808x^9 - 68147233556x^8 - 184773020492x^7 - 262836346620x^6 - 216526023556x^5 - 122659285853x^4 - 39783078178x^3 + 1029344695x^2 + 7429526904x + 2484513522)$, *and* $a_0 = 4096(x+1)^2(x+2)^2(2x+3)^2(2x+1)^2(4x+33)(524160x^8 + 13584480x^7 - 37764552x^6 - 736049716x^5 - 3014273813x^4 - 6181409598x^3 - 7122549901x^2 - 4430333096x - 1162363872)$.

First we check if L is reducible by computing factors of orders $d = 1, 2, 3$. Setting `infolevel[LREtools]:=10` *causes* `RightFactors` *([25] and [4, Section 7]) to display every candi-*

[2]The CPU time depends on the degree bound for P, computed for each λ_i in ([4, Section 5] and [50, Section 4.5]). However, we have no reasonable a priori degree bound.

date it checks, plus some other information. For $d = 2$ it checks 30 candidates in order to find all hypergeometric solutions of the 6 by 6 system $\bigwedge^d(C_L)$. This means it computes polynomial solutions of 30 systems. Each of these is converted to a system over $\mathbb{Q}$ using degree bounds computed from generalized exponents [50, Chapter 6]. After that, the number of unknowns in $\mathbb{Q}$ of these systems ranges from 13 to 43, and the number of equations ranges from 74 to 117. Most of the 30 systems have no solutions, and the remaining ones have solutions that do not correspond to factors. All of this takes only a fraction of a second and we find that L is irreducible in D.

Factoring $L^{(2)}$ in Step 1(c) takes much longer, primarily because the number of candidate types is much higher for $L^{(2)}$ than it is for L (the order is still 4 but the degree in x is much higher). This time `RightFactors` *reports 1791 candidate combinations (see Remark 4.2.2 below). Degree bounds and the resulting systems over $\mathbb{Q}$ are larger too.*

Remark 4.2.2. *One technical point in the implementation* `RightFactors` *is that it constructs more than the candidate types explained in [50, 4]. It constructs* candidate combinations *for the factors R to be computed. A combination for R contains not only the type (i.e.* $\det(R)$ *up to $\sim$) but also the Newton polygon and polynomials of R. (For a definition see [10]). Due to a relation between these ingredients, the number of candidate combinations could be less than the number of candidate types, but it could also be greater, because one candidate type could match more than one Newton polygon plus polynomials. But the general situation remains that the CPU time for RightFactors depends greatly on the number of candidates. The purpose of Theorem 4.3.1 below is to discard candidates when possible.*

4.3 Efficiency Improvement of Absolute Factorization

Theorem 4.3.1. *Let $L \in D$ be irreducible of order n, and let p be a prime dividing n. Suppose that $L^{(p)}$ has order n as well. Then $\psi_p(\det(R)) \sim \det(L)$ for any factor R of $L^{(p)}$ of order n/p.*

By passing this information to RightFactors in Step 1(c) on algorithm `AbsFactorization`, the number of candidate combinations in Example 4.2.1 reduces from 1791 to 363, resulting in a significant improvement in running time, from 24.7 to 6.9 seconds. The reason the number of cases is not 1 is because $\psi_p(\lambda_1) \sim \psi_p(\lambda_2)$ does not imply $\lambda_1 \sim \lambda_2$.

Our algorithm is also an efficient way to compute Liouvillian solutions, which corresponds to order 1 factors of $L^{(n)}$.

Proof of Theorem 4.3.1: When L is irreducible and $L^{(p)}$ is not, we have $D/DL \cong D/D\tilde{L}$ for some $\tilde{L} \in D_p$, see Theorem 4.6.1. Now $D/D\tilde{L}$ is an irreducible D-module with basis $B = \{\Phi^0, \ldots, \Phi^{n-1}\}$. Let $M_i := \mathrm{SPAN}_{\mathbb{C}(x)}(B_i) \subset D/D\tilde{L}$ where $B_i := \{\phi^j \in B \mid j \equiv i \mod p\}$. This M_i is a D_p-module because $\tilde{L}$ is in D_p. Now $\tilde{L}_i := \tilde{L}|_{x \mapsto x+i} = \Phi^i \tilde{L} \Phi^{-i} \in D_p$ annihilates $\Phi^i \in M_i \subset D/D\tilde{L}$ and hence $M_i \cong D_p/D_p\tilde{L}_i$. ($\Phi^i \tilde{L} \Phi^{-i}$ is computed in the larger ring $\mathbb{C}(x)[\Phi, \Phi^{-1}] \supseteq D$). We find

$$(D/DL)\downarrow_p^1 \;\cong\; (D/D\tilde{L})\downarrow_p^1 \;=\; \bigoplus_{i=0}^{p-1} M_i \;\cong\; \bigoplus_{i=0}^{p-1} \left(D_p/D_p\tilde{L}_i \right).$$

If N is a D_p-module then we can use the isomorphism $\psi_p : D \to D_p$ to construct a D-module $[N] := \{[n] \mid n \in N\}$ by defining $G \cdot [n] := [\psi_p(G) \cdot n]$ for any $G \in D$.

The minimal operator of $[1] \in [(D/DL)\downarrow_p^1]$ is $L^{(p)}$. We assumed $\mathrm{order}(L^{(p)}) = n$, which implies $D/DL^{(p)} \cong [(D/DL)\downarrow_p^1]$, because otherwise the algorithm stops in Step 1(b). So

$$D/DL^{(p)} \;\cong\; [(D/DL)\downarrow_p^1] \;\cong\; \bigoplus_{i=0}^{p-1} [D_p/D_p\tilde{L}_i] \;\cong\; \bigoplus_{i=0}^{p-1} (D/DL_i)$$

where $L_i := \psi_p^{-1}(\tilde{L}_i)$ is irreducible in D (because $\tilde{L}$ and $\tilde{L}_i$ are irreducible in D_p, and even irreducible in D). This shows that if R is any factor of $L^{(p)}$ of order n/p, then $D/DR \cong D/DL_i$ and $\det(R) \sim \det(L_i)$ for some i. Then $\psi_p(\det(R)) \sim \det(\tilde{L}_i) \sim \det(\tilde{L}) \sim \det(L)$.

Remark 4.3.2. *(continuation of Remark 4.2.2) As mentioned in Remark 4.2.2, our factoring implementation constructs not only candidates for* $\det(R)$ *up to* $\sim$*, but it combines them with candidates for the Newton polygon and polynomials for* R*. But we know the latter data ahead of time. This is because gauge equivalent operators* L *and* $\tilde{L}$ *have the same Newton polygon and polynomials, and* $\psi_p^{-1}(\tilde{L})$ *gives this data for* R*. Combinations with the wrong Newton polygon or polynomials can thus be discarded, which further reduces the number of cases from 363 to 121, and the CPU time from 6.9 to 3.2 seconds.*

In total we have a reduction from 24.7 to 3.2 seconds for the absolute factorization of L from Example 4.2.1. The improvement coming from Theorem 4.3.1 will be smaller for examples where a_0, a_n

have fewer factors, because if there are fewer combinations then there is less room for improvement. See [24] for more examples.

4.4 Example: OEIS A260772

In the OEIS database, the sequence $A260772$

$$(1, 3, 10, 41, 190, 946, \dots) \in \mathbb{C}^{\mathbb{N}} \tag{4.1}$$

counts certain directed lattice paths. It has the minimal operator

$$\begin{aligned} L \coloneqq &(x+5)(x+4)(25x^2+130x+141)\Phi^4 - 30(x+4)(7x+13)\Phi^3 \\ &- (1100x^4+12320x^3+48664x^2+80740x+47400)\Phi^2 \\ &+ 120(x+6)(x+1)\Phi - 16x(x+1)(25x^2+180x+296). \end{aligned}$$

Even though L is irreducible of order 4, it is nevertheless 2-solvable, as we now show. The subsequence of (4.1),

$$u(n) = A260772(2n) = (1, 10, 190, \dots) \in \mathbb{C}^{\mathbb{N}}$$

is called a *2-section* of (4.1). The other 2-section is $A260772(2n+1) = 3, 41, 946, \dots$

The minimal recurrence of $u(n)$ is the *2-section operator* from Definition 3.3.1:

$$\begin{aligned} L^{(2)} = &(4x^4+56x^3+287x^2+634x+504)\Phi^4 \\ &+ (-352x^4-4048x^3-17276x^2-32354x-22344)\Phi^3 \\ &+ (7616x^4+68544x^3+229648x^2+339408x+186648)\Phi^2 \\ &+ (5632x^4+36608x^3+86336x^2+88288x+32928)\Phi \\ &+ 1024x^4+4096x^3+4352x^2+1280x. \end{aligned}$$

But this operator is reducible! Namely,

$$L^{(2)} = \mathrm{LCLM}(R, R') \tag{4.2}$$

where R and R' are irreducible second order operators given in (4.6) below. This means that the 2-section $u(n)$ of A260772 is 2-expressible. Equation (4.2) expresses the fact that $u(n)$ can be written as

$$u(n) = t(n) + t'(n) \tag{4.3}$$

for some $t,\ t' \in \mathbb{C}^{\mathbb{N}}$ that satisfy $R,\ R'$. Computing bases of solutions of R and R' in $\mathbb{C}^{\mathbb{N}}$ and comparing initial conditions gives $t(0) = 0, t(1) = 4$, and $t'(0) = 1, t'(1) = 6$. The recurrence $R(t) = 0$ reads:

$$t(x+2) = \frac{a_0 t(x) + a_1 t(x+1)}{(x+2)(2x+5)(5x+3)} \tag{4.4}$$

which expresses $t(2), t(3), \ldots$ in terms of prior t values. Here $a_0 = 16x(2x+1)(5x+8)$ and $a_1 = 440x^3 + 1584x^2 + 1780x + 600$.

We can express $t'(n), n \in \mathbb{N}$ by using (4.4) for $x \in -\frac{1}{2} + \mathbb{N}$, as follows. Let $t(-\frac{1}{2}) = -\frac{1}{2}$ and $t(\frac{1}{2}) = 1$, and define $t(x+2)$ for $x \in -\frac{1}{2} + \mathbb{N}$ with (4.4). By computing a gauge transformation from $V(R|_{x \mapsto x+\frac{1}{2}})$ to $V(R')$, we find

$$t'(n) = \frac{(2-4n)t\big(n-\frac{1}{2}\big) + (2n+2)t\big(n+\frac{1}{2}\big)}{1+10n}. \tag{4.5}$$

See [24] for further details.

Recall that $u(n) = A260772(2n)$. Substituting (4.5) and $n \mapsto n/2$ in (4.3) gives a formula (that we uploaded to OEIS) for $A260772(n)$, written in terms of solutions (in $\mathbb{C}^{\mathbb{N}}$ and in $\mathbb{C}^{-\frac{1}{2}+\mathbb{N}}$) of a *second order* recurrence (4.4).

This example illustrates one of the cases in the classification of 2-solvable order 4 equations. Algorithm `AbsFactorization` covers this case, which we will illustrate with the same example.

L has order 4 so the only prime p in Step 1 is $p = 2$. Step 1a computes $L^{(2)}$. It has order 4 (Step 1b) so we proceed with Step 1c which computes the set of right factors $S = \{R, R'\}$ where

$$\begin{aligned} R &= (2x+5)(5x+3)(x+2)\Phi^2 \\ &-(440x^3 + 1584x^2 + 1780x + 600)\Phi - 8(5x+8)(4x^2+2x), \\ R' &= (2x+5)(10x+9)(x+2)\Phi^2 \\ &-(880x^3 + 3432x^2 + 4220x + 1650)\Phi - 16(10x+19)(2x^2+x). \end{aligned} \tag{4.6}$$

R and R' generate nontrivial submodules of $D/DL^{(2)}$. Applying ψ_2 gives generators of the corresponding submodules of $(D/DL)\downarrow_2^1$.

4.5 Absolute Order

The right factors R and R' in (4.6) exemplify a general phenomenon: for an irreducible operator L, all irreducible factors of $L^{(m)}$ will have the same order. This is a consequence of Corollary 4.5.2 below, using the concept of absolute order which we now define.

Definition 4.5.1. *Let M be a D-module. Let the sequence*

$$0 = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_{k-1} \subsetneq M_k = M$$

of D-submodules be a composition series of M, i.e., for each i, M_{i+1}/M_i is irreducible. Then

1. $\mathrm{Orders}_D(M)$ *is defined as the sorted list of numbers* $\dim_{\mathbb{C}(x)}(M_{i+1}/M_i)$.
2. $\mathrm{AbsOrders}(M)$ *is defined as* $\mathrm{Orders}_{D_t}(M{\downarrow_t^1})$ *for a value of t for which $M{\downarrow_t^1}$ has the maximum number of composition factors.*

Note that $\mathrm{Orders}_D(M)$ is an ordered partition of $\dim_{\mathbb{C}(x)}(M)$ and independent of the choice of composition series by the Jordan-Hölder theorem. Likewise, $\mathrm{AbsOrders}(M)$ is well-defined: two composition series for $M{\downarrow_s^1}$ and $M{\downarrow_t^1}$ maximizing the number of composition factors must restrict to two composition series of $M{\downarrow_{st}^1}$ by the maximality assumption (Any D_s-module is also a D_{st}-module). By the Jordan-Hölder theorem, their composition factors coincide, so $\mathrm{Orders}_{D_s}(M) = \mathrm{Orders}_{D_t}(M)$.

Corollary 4.5.2 (of Theorem 4.6.1 below)**.** *If M is an irreducible D-module of dimension n, then*

1. $\mathrm{AbsOrders}(M) = [m, \ldots, m]$ *(k copies of m) for some k, m with $km = n$. Moreover, for any t we have:*
2. $\mathrm{Orders}_{D_t}(M) = [m, \ldots, m]$ *(k copies of m) if and only if $k \mid t$.*

Proof.

(1): If M is absolutely irreducible, then $m = n$ and $k = 1$. Otherwise, $M{\downarrow_p^1}$ is reducible for some prime factor p of n by Theorem 4.0.2. By Theorem 4.6.1, $M{\downarrow_p^1} \cong N \oplus \Phi(N) \oplus \cdots \oplus \Phi^{p-1}(N)$. After

applying the isomorphism in Equation (3.3), we invoke induction to conclude that $\text{AbsOrders}(N) = [m, \ldots, m]$ (k' copies of m) for some m, k' with $k'm = n/p$. Therefore,

$$\text{AbsOrders}(N \oplus \Phi(N) \oplus \cdots \oplus \Phi^{p-1}(N)) = [m, \ldots, m] \quad (k \text{ copies of } m)$$

with $k = pk'$.

(2): Argue as in (1). □

Corollary 4.5.2 also follows from Proposition C.0.2 .

4.6 Second Absolute Factorization Algorithm

Implementation is in [24]. Let ζ_p be a primitive pth root of unity.

Algorithm: `AbsIrreducibility`

Input: An irreducible operator $L \in D$.[3] **Output:** `True` if L is absolutely irreducible; otherwise `False`.

1. For each prime $p \mid \text{order}(L)$:
   (a) Compute $\widetilde{L} \coloneqq L|_{\Phi \mapsto \Phi/\zeta_p}$ which equals $L \circledS (\Phi - \zeta_p)$.
   (b) Compute the set $\text{Hom}_D(D/D\widetilde{L}, D/DL)$ using [4, 23].
   If this set is nontrivial, return `False`.
2. Return `True`.

Correctness of this algorithm follows from

Theorem 4.6.1. *Let L be irreducible in D. Take $M = D/DL$ and let p be a prime. Then the following are equivalent:*

1. *L is gauge equivalent to an element of D_p;*
2. *L is gauge equivalent to $L \circledS (\Phi - \zeta_p)$;*

[3] Factoring is explained in [4, 50]. The Maple command `RightFactors`, implemented by van Hoeij, factors L if it is reducible. By default, it factors in $C(x)[\phi]$ where C is the smallest field of constants over which the input is defined. To ensure a full factorization over $\mathbb{C}$ one may need to add field extensions to an optional input of `RightFactors` to ensure that C is large enough.

3. $M \cong M \otimes D/D(\Phi - \zeta_p)$ *as* D*-modules;*

4. $M{\downarrow_p^1}$ *is reducible.*

5. $M{\downarrow_p^1} \cong N \oplus \Phi(N) \oplus \cdots \oplus \Phi^{p-1}(N)$ *for some irreducible* D_p*-submodule* N *of* $M{\downarrow_p^1}$.

Proof.

(1) $\Rightarrow$ (2): Suppose that L is gauge equivalent to $L' \in D_p$. Then L is also gauge equivalent to $L \circledS (\Phi - \zeta_p)$ since $L' = L' \circledS (\Phi - \zeta_p)$.

(2) $\Rightarrow$ (3): $L \circledS (\Phi - \zeta_p)$ is the minimal operator for $1 \otimes 1$ in $M \otimes D/D(\Phi - \zeta_p)$.

(3) $\Rightarrow$ (4): $D/D(\Phi - \zeta_p)$ and $D/D(\Phi - 1)$ are isomorphic as D_p-modules but not as D-modules. The composite D_p-module isomorphism

$$M \cong M \otimes D/D(\Phi - \zeta_p) \cong M \otimes D/D(\Phi - 1) \cong M$$

is therefore nontrivial. So $M{\downarrow_p^1}$ is reducible by Lemma 2.0.10.

(4) $\Rightarrow$ (5): Since the prime p has no proper divisors, by Theorem 3.4.4, $M \cong D \otimes_{D_p} N$ for some irreducible D_p-module N. Now use the fact that D is a free left D_p-module with basis $1, \Phi, \Phi^2, \ldots, \Phi^{p-1}$.

(5) $\Rightarrow$ (1): By the cyclicity of the direct sum decomposition, a cyclic vector u of N is also cyclic for M with minimal operator in D_p. $\square$

If L is not absolutely irreducible, we can find a factorization by adjusting `AbsIrreducibility` with the proof of Theorem 3.4.4. We illustrate this for $p = 2$.

Suppose $\mathrm{Hom}_D(D/D\widetilde{L}, D/DL)$ from Step 1b has a nonzero element G. Then nontrivial factors of $L{\downarrow_2^1}$ in D_2 are constructed as follows:

1. Let c be the remainder of $\widetilde{G}G \pmod{DL}$ where $\widetilde{G} = G|_{\phi \mapsto -\phi} \in \mathrm{Hom}_D(D/DL, D/D\widetilde{L})$. Since L is irreducible, c must be a constant by Lemma 2.0.10. (This c is the same as in the proof of Theorem 3.4.4).

2. Replace G with $G/\sqrt{c}$.

3. Now $1+G$ resp. $1-G$ are maps from $V(L)$ to $V(L)+V(\tilde{L}) = V(L{\downarrow}_2^1)$. The images of these maps are solution spaces of right-factors of $L{\downarrow}_2^1$. Remark 2.0.13 shows how to compute them, which, due to $\Phi \mapsto -\Phi$ symmetry, will be in D_2. This way the Hom computation in Step 1(b) replaces a call to `RightFactors`.

This $1+G$ is the same as Ψ in the proof of Theorem 3.4.4. For $p > 2$, `AbsIrreducibility` and its extension above require the use of the field extension $\mathbb{Q}(\zeta_p)/\mathbb{Q}$. This may impact the efficiency of `AbsIrreducibility` compared to that of `AbsFactorization`.

If $p = \mathrm{ord}(L)$ then `AbsFactorization` becomes equivalent to computing Liouvillian solutions, for which improvements were given in [11].

CHAPTER 5

ORDER 3 ALGORITHM

This chapter is reproduced from joint work [6].

As explained in the introduction, to solve third order equations in terms of second order equations, it suffices to cover cases (R), (L) and (S^2). Cases (R) and (L) are handled by (absolute) factorization, so we focus on case (S^2). This can be broken down into two steps:

1. Find, if it exists, a gauge transformation G from L_3 to a third order operator L_G for which $\operatorname{order}(L_G^{\circledS 2}) = 5$. This is done in Section 5.2.
2. Decompose L_G as $L_2^{\circledS 2} \circledS (\Phi - r)$ using case (c) in Theorem 5.1.1(3). (Cases (a) and (b) in Theorem 5.1.1(3) are already covered by cases (L) and (R)).

5.1 Simpler Case

The easiest sub-case of case (S^2) is when that gauge-equivalence is simply an equality: $L_3 = L_2^{\circledS 2} \circledS (\Phi - r)$. This section shows how to detect this sub-case, and if so, how to find L_2 and r.

Theorem 5.1.1. *Let $L_3 = \Phi^3 + c_2\Phi^2 + c_1\Phi + c_0$ be a third order difference operator over $\mathbb{C}(x)$. The following are equivalent.*

1. *The element $I(c_0, c_1, c_2)$ given by*

$$\begin{aligned} &+ \Phi^2 c_0 \cdot c_1 \cdot c_2 \cdot (\Phi c_2)^2 \cdot \Phi^2 c_2 \quad -\Phi^2 c_0 \cdot c_1 \cdot \Phi^2 c_1 \cdot c_2 \cdot \Phi c_2 \\ &- \Phi^2 c_0 \cdot c_2 \cdot \Phi c_2 \cdot \Phi c_0 \cdot \Phi^2 c_2 \quad - (\Phi c_1)^2 \cdot c_1 \cdot \Phi^2 c_1 \cdot \Phi^2 c_2 \\ &+ \Phi c_0 \cdot \Phi c_1 \cdot \Phi^2 c_1 \cdot c_2 \cdot \Phi^2 c_2 \qquad +\Phi^2 c_0 \cdot \Phi c_1 \cdot c_1 \cdot \Phi^2 c_1 \end{aligned} \tag{5.1}$$

   is zero in $\mathbb{C}(x)$.

2. *The operator $L_3^{\circledS 2}$ has order ≤ 5.*
3. *One of the following three cases holds.*

(a) $L_3 = \Phi^3 + c_0$, *with* $c_0 \neq 0$;

(b) $L_3 = (\Phi + c_2) \circ (\Phi^2 + \Phi^{-1}(c_1))$;

(c) Let $L_2 = \Phi^2 + a\Phi + b$. *For some* $a, r \in \mathbb{C}(x)^*$ *and* $b \neq 1$, *we have*

$$L_3 = L_2^{\circledS 2} \circledS (\Phi - r). \tag{5.2}$$

In this case we may choose

$$a = 1, \quad b = \frac{\begin{vmatrix} 1 & \frac{\phi^{-1}(c_2)\, c_2}{c_1} \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} 1 & \frac{\phi^{-1}(c_2)\, c_2}{c_1} \\ \frac{c_1\, \phi^{-1}(c_1)}{c_0\, \phi^{-1}(c_2)} & 1 \end{vmatrix}}, \quad r = \frac{\phi^{-2}(c_2)}{\phi^{-1}(b) - 1}.$$

Proof. A computer verification shows the Φ^6-coefficient of $L_3^{\circledS 2}$ is $I(c_0, c_1, c_2)$, so (1) and (2) are equivalent. We now take the three special cases and study them.

Case (a). Suppose c_1 or c_2 is zero. By inspection, we see that I is zero precisely when $c_1 = c_2 = 0$, i.e., when $L_3 = \Phi^3 + c_0$.

Case (b). Suppose $c_0 = \phi^{-1}(c_1)c_2$. Then $I = 0$ and L_3 factors as shown.

Case (c). Suppose that $c_1, c_2 \neq 0$ and $c_0 \neq \phi^{-1}(c_1)c_2$. We consider when (5.2) holds. Now $a \neq 0$, otherwise the right side of (5.2) would have order 2. Replacing L_2 with

$$L_2 \circledS (\Phi - 1/a) = \Phi^2 + \Phi + b'$$

and multiplying r by a^2 does not change the right side of (5.2), so we may further assume $a = 1$.

Matching coefficients in (5.2) now gives a system of equations

$$\begin{aligned} [\Phi^2]: &\quad c_2 = \phi^2(r)[\phi(b) - 1], \\ [\Phi]: &\quad c_1 = -\phi(r)\phi^2(r)\phi(b)[\phi(b) - 1], \\ [1]: &\quad c_0 = -r\,\phi(r)\phi^2(r)\, b^2\, \phi(b). \end{aligned} \tag{5.3}$$

Since $c_1, c_2 \neq 0$, (5.3) is equivalent to

$$\begin{aligned} [\Phi^2]: &\quad c_2 = \phi^2(r)[\phi(b) - 1], \\ \frac{\phi^{-1}[\Phi^2] \cdot [\Phi^2]}{[\Phi]}: &\quad \frac{\phi^{-1}(c_2)\, c_2}{c_1} = \frac{1 - b}{\phi(b)}, \\ \frac{[\Phi] \cdot \phi^{-1}[\Phi]}{[1] \cdot \phi^{-1}[\Phi^2]}: &\quad \frac{c_1\, \phi^{-1}(c_1)}{c_0\, \phi^{-1}(c_2)} = \frac{1 - \phi(b)}{b}. \end{aligned} \tag{5.4}$$

The last two equations can be written as the linear system

$$A\begin{bmatrix} b \\ \phi(b) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & \dfrac{\phi^{-1}(c_2)\, c_2}{c_1} \\ \dfrac{c_1\, \phi^{-1}(c_1)}{c_0\, \phi^{-1}(c_2)} & 1 \end{bmatrix}.$$

Since $c_0 \neq \phi^{-1}(c_1)c_2$, we have $\det A \neq 0$. Cramer's rule now gives

$$b = \frac{1}{\det(A)} \begin{vmatrix} 1 & \frac{\phi^{-1}(c_2)\, c_2}{c_1} \\ 1 & 1 \end{vmatrix}, \quad \phi(b) = \frac{1}{\det(A)} \begin{vmatrix} 1 & 1 \\ \frac{c_1\, \phi^{-1}(c_1)}{c_0\, \phi^{-1}(c_2)} & 1 \end{vmatrix}. \tag{5.5}$$

Consistency of (5.3) is now equivalent to the condition $\phi(f) - g = 0$, where f and g are the right sides in (5.5). But the numerator of $\phi(\phi(f)-g)$, written as a rational function of $\phi^i(c_j)$, is precisely $I(c_0, c_1, c_2)$. Therefore, (5.2) holds precisely when $I(c_0, c_1, c_2) = 0$.

Finally, the expression for b appearing in the statement of this theorem is f here as we mentioned before, and r is written in terms of b via the first equation in (5.3). □

5.2 Algorithm

5.2.1 The Gauge Transformation G

Consider an irreducible third order operator $L_3 \in \mathbb{C}(x)[\Phi, \Phi^{-1}]$. For any nonzero $G = b_0 + b_1\Phi + b_2\Phi^2 \in \mathbb{C}(x)[\Phi, \Phi^{-1}]$, $G(V(L_3))$ is the solution space of some third order operator that we will denote as L_G. Then $L_G G$ is right divisible by $\mathrm{LCLM}(L_3, G)$ and hence equals it (by comparing orders). Therefore L_G equals $\mathrm{LCLM}(L_3, G)$ right-divided by G.

In the algorithm below, we initially do not know the correct values of b_0, b_1, b_2 in $\mathbb{C}(x)$. Therefore in Step 5, b_0, b_1, b_2 are first viewed as variables, and G is in $\mathbb{C}(x)[b_0, b_1, b_2][\Phi, \Phi^{-1}]$. Now G induces a map

$$G_2 : V(L_3^{\circledS 2}) \to V(L_G^{\circledS 2}). \tag{5.6}$$

We can write $G_2 = \sum_{i=0}^{5} a_i \Phi^i$ with $a_i \in \mathbb{C}(x)$. This G_2 needs to meet the following requirement: If $u \in V(L_3)$, then $u^2 \in V(L_3^{\circledS 2})$, and with G sending u to $G(u)$, we need G_2 to send u^2 to $G(u)^2$. The latter is a quadratic polynomial in b_0, b_1, b_2 over R, where R is defined to be $\mathbb{C}(x)[u^2, u\phi(u), \ldots, \phi^2(u)^2]$.

The inclusion

$$\mathbb{C}(x)[u^2, \phi(u)^2, \ldots, (\phi^5(u))^2] \subseteq R$$

is computed in Step 4 of the algorithm below. It allows us to write $G_2(u^2)$ as an R-linear combination of $a_0, \ldots, a_5$. We then equate that to $G(u)^2$, which is is an R-linear combination of $b_0^2, b_0b_1, \ldots, b_2^2$. Taking coefficients with respect to $u^2, u\phi(u), \ldots, \phi^2(u)^2$ gives six $\mathbb{C}(x)$-linear relations between the a_i and $b_0^2, b_0b_1, \ldots, b_2^2$. Solving those writes each a_i as a quadratic polynomial in b_0, b_1, b_2 over $\mathbb{C}(x)$. In this way, we obtain an expression for G_2 in $\mathbb{C}(x)[b_0, b_1, b_2][\Phi, \Phi^{-1}]$ that is quadratic in b_0, b_1, b_2.

If (5.6) is not injective, then its image has dimension ≤ 5 which is the case handled in Section 5.1. That relates L to an operator L_G for which Theorem 5.1.1(2) applies. So we need a subspace $V(L_1) \subset V(L_3^{\mathbb{S}^2})$ in the kernel of G_2. This means that the remainder of G_2 right-divided by L_1 needs to be 0. This L_1 will have order 1, and the remainder will be a quadratic polynomial in b_0, b_1, b_2 over $\mathbb{C}(x)$. Next we have to find a nontrivial solution for that quadratic polynomial (i.e., find a point on a conic over $\mathbb{C}(x)$). Substituting that solution into G ensures that this remainder is zero, so that (5.6) is not injective.

5.2.2 Algorithm Steps

Implementation is in [24].

Algorithm: `ReduceOrder`

Input: An absolutely irreducible third order operator $L_3 \in D$.

Output: $(G, \widetilde{G}, L_2, r)$ if L_3 is 2-solvable, where G is a gauge transformation, $\widetilde{G}$ is the inverse of G and sends L_3 to an operator of the form (5.2), and L_2 and r are from (5.2). Otherwise: `Not 2-solvable`.

1. $L_6 \coloneqq L_3^{\mathbb{S}^2}$.
2. If $\text{order}(L_6) = 5$, then return $G = 1$, $\widetilde{G} = 1$ and return L_2, r from Theorem 5.1.1 (3) (c).
3. Now $\text{order}(L_6) = 6$. If it exists, let L_1 be an order-1 right-factor of L_6. Otherwise: return `Not 2-solvable`.
4. Let u be a symbolic solution of L_3, which means that $u(x), u(x+1), u(x+2)$ are symbols, and that we write $u(x+i), i \geq 3$ as $\mathbb{C}(x)$-linear combinations of $u(x), u(x+1), u(x+2)$ in such a way that $L_3(u) = 0$.

5. Let $G = b_0+b_1\Phi+b_2\Phi^2$ and $G_2 = a_0\Phi^0+\cdots+a_5\Phi^5$ where the a_i and b_i are new variables. Now compute $G_2(u(x)^2)$ and write it as a $\mathbb{C}(x)[a_0,\ldots,a_5]$-linear combination of $u(x)^2, u(x)u(x+1),\ldots,u(x+2)^2$.

6. Compute $G_2(u(x)^2) - (G(u(x)))^2$. Let S be the set of its 6 coefficients as a quadratic polynomial in $u(x), u(x+1), u(x+2)$.

7. Solve S to write $a_0,\ldots,a_5$ as quadratic polynomials in b_0, b_1, b_2 over $\mathbb{C}(x)$. Substitute this in G_2 so that $G_2 \in \mathbb{C}(x)[b_0,b_1,b_2][\Phi,\Phi^{-1}]$.

8. Let $C \in \mathbb{C}(x)[b_0,b_1,b_2]$ be the remainder of G_2 right-divided by L_1. This C is a *conic* (quadratic homogeneous polynomial in 3 variables).

9. Simplify conic C with linear transformations to a conic of the form $c_1X_1^2 + c_2X_2^2 + c_3X_3^2 \in \mathbb{C}(x)[X_1,X_2,X_3]$.

10. Apply a conic-solver [19] to find a nontrivial point $(X_1,X_2,X_3) \in \mathbb{C}(x)^3$. Reverse the transformations to find a point $(b_0,b_1,b_2) \in \mathbb{C}(x)^3$ on C.

11. Substitute the point into G to obtain $G \in \mathbb{C}(x)[\Phi,\Phi^{-1}]$.

12. Compute L_G by right-dividing $\mathrm{LCLM}(L_3, G)$ by G.

13. As explained in Section 5.2.1, the symmetric square of L_G has order 5 and so we can use Theorem 5.1.1 to write it as $L_2^{\mathbb{S}2} \circledS (\Phi - r)$ for some $L_2 \in D$ and $r \in \mathbb{C}(x)$.

14. Compute $\widetilde{G}$, the inverse gauge transformation of G, with the extended Euclidean Algorithm (see Remark 2.0.13).

15. Return $G, \widetilde{G}, L_2, r$ and stop.

Remark 5.2.1. *If $(G_a, \tilde{G}_a, L_{2a}, r_a)$ and $(G_b, \tilde{G}_b, L_{2b}, r_b)$ are two possible outputs, then L_3 is gauge-equivalent to both $L_{2a}^{\mathbb{S}2}\circledS(\Phi-r_a)$ and $L_{2b}^{\mathbb{S}2}\circledS(\Phi-r_b)$. Hence $L_{2a}^{\mathbb{S}2}$ and $L_{2b}^{\mathbb{S}2}$ are projectively equivalent. Because ρ_3 from Definition 3.7.7 has kernel $\{\pm1\} \subset \mathbb{C}^*$, this implies (Proposition 2.3.7) that L_{2a} and L_{2b} are projectively equivalent as well. Thus, the L_2 in the output is unique up to projective equivalence. For a similar result in the differential case, [22, Theorem 4.7].*

Our most recent implementation, `ReduceToOrderTwo` *at [24] now uses an algorithm* `NormalForm` *to simplify the output. This algorithm is based on Chapter 5 in [5].*

Given an irreducible operator L_2, it aims to find a projectively equivalent operator of near optimal size.

5.3 Examples

See [24] to download these examples and the code.

5.3.1 Example: OEIS A295371

Let

$$a(n) = \frac{1}{2n}\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}\binom{2k}{k}(k+2)(-3)^{(n-1-k)}.$$

Zhi-Wei Sun conjectured that $a(n)$ is a positive odd integer for all $n > 0$. We prove[1] this conjecture by using Algorithm `ReduceOrder` to find a formula for $a(n)$ without n in the denominator. OEIS lists the recurrence

$$L_3 = (2x+1)(x+3)^2\Phi^3 - (2x+1)(7x^2+38x+52)\Phi^2 \\ - 3(2x+5)(7x^2+4x+1)\Phi + 27(2x+5)x^2$$

for $a(n)$. Algorithm `ReduceOrder` returns the pair $L_2, \Phi - r$ given by

$$\Phi^2 + \Phi - \frac{3(x+1)}{4(x-1)},$$
$$\Phi - \frac{4(x^2+3x+3)(2x+5)(x-1)^2}{(x^2+x+1)(2x+3)x(x+1)},$$

as well as the gauge transformations G and $\widetilde{G}$, omitted here, between the solution spaces of L_3 and $L_2^{\circledS 2} \circledS (\Phi - r)$.

Next, an implementation from [9] solves L_2 in terms of another OEIS entry. After simplifying, we find

$$a(n) = (b(n)^2 + 3b(n-1)^2)/4,$$

where $b(n)$ is the OEIS entry A002426. To verify that no errors were introduced during the simplification or other steps, we can prove this formula by computing recurrences for $a(n)$ and $b(n)$, and comparing the recurrences and initial values. Analyzing $b(n) \pmod 2$ proves that $a(n)$ is an odd integer.

[1] A referee in my paper [6] pointed out that Lemma 2.3 in [48] proves a closely related statement where $a(n)$ has $\binom{n+1}{k}$ instead of $\binom{n-1}{k}$.

Solving equations in closed form, or, as in this paper, solving equations in terms of solutions of other equations, is often helpful for proofs. This is not limited to difference equations; one can also prove that $a(n) \in \mathbb{Z}$ by computing closed form solutions of the differential equation for the generating function.

5.3.2 Example: OEIS A178808

Let

$$a(n) = \frac{1}{n^2} \sum_{k=0}^{n-1} (2k+1)(D_k)^2,$$

where $D_0, D_1, \cdots$ are central Delannoy numbers given by A001850.

Zhi-Wei Sun conjectured that $a(n)$ is always an integer and proved it in arXiv:1008.3887.

Our algorithm leads to this formula

$$a(n) = \frac{6c(n)c(n-1) - c(n)^2 - c(n-1)^2}{8}$$

where $c(n)$ is the OEIS entry A001850. We can use it to prove the conjecture.

5.3.3 Example: OEIS A268138

Let

$$a(n) = \frac{1}{n} \sum_{k=0}^{n-1} A001850(k) A001003(k+1).$$

The OEIS tells us that Zhi-Wei Sun conjectured that $a(n)$ is an odd integer, and essentially proved it in arXiv:1602.00574. It can also be proved from the following formula produced with our algorithm

$a(n) =$

$$\frac{n(n+1)c^2(n) + (n-1)^2 c^2(n-1) - 3(n-1)(2n+1)c(n)c(n-1)}{2}$$

where $c(n)$ is the OEIS entry A001003. We can use it to prove the conjecture.

CHAPTER 6

ORDER 4 ALGORITHMS

This chapter is reproduced from joint work [8].

6.1 Case (3): Symmetric Product

If an irreducible operator L of order 4 is gauge equivalent to a symmetric product of second order operators, then $\bigwedge^2(L)$ has two right-factors of order three, both 2-solvable (see Appendix G, or [38, Section 4.4] for the differential case).

We divide Case(3) into two sub-cases: Case 3(a) is when L itself is gauge equivalent to a symmetric product, and Case 3(b) is when $L^{(2)}$, but not L, is gauge equivalent to a symmetric product. Theorem 4.3.1 provides a significant efficiency improvement for Case 3(b). The implementation can be found at [24].

Algorithm: `Case 3(a)`

Input: An irreducible order 4 operator L.

Output: If the exterior square of L does not have order 3 right factors then return FAIL, otherwise return two order 2 operators L_{2a} and L_{2b} and a gauge transformation $G : V(L_{2a} \circledS L_{2b}) \to V(L)$ (see Remark 6.1.1 and Definition 2.0.12 for the definition of gauge transformation).

1. Let $L_6 := \bigwedge^2(L)$ (see Section 2.2).
2. Let $S = \text{RightFactors}(L_6, 3)$. If $S = \emptyset$ then return FAIL.
   Otherwise S should have two elements, say L_{3a} and L_{3b}.
3. Apply the ReduceOrder algorithm (in Chapter 5) to L_{3a} and L_{3b} to obtain L_{2a} and L_{2b} of order 2. If this fails then return FAIL.
4. $L_s := L_{2a} \circledS L_{2b}$.

5. Find an operator G that maps $V(L_s)$ to $V(L)$, see Remark 6.1.1.

6. Return L_{2a}, L_{2b} and G.

Remark 6.1.1. *As mentioned in Remark 5.2.1, the operators L_{2a}, L_{2b} are only unique up to projective equivalence. So we are free to replace say L_{2a} by $L_{2a} \circledS (\Phi - r)$ for any non-zero r, which would replace L_s by $L_s \circledS (\Phi - r)$. This would change $\det(L_s)$ by a factor r^4. This r needs to be chosen in the right way, because a necessary condition for a gauge transformation G from L_s to L to exist is that $\det(L_s) \sim \det(L)$. This condition determines r up to $\sim$ and up to a factor in $\{1, i, -1, -i\}$. For each of these 4 cases (2 cases if we restrict to working over $\mathbb{Q}$) we have to compute a gauge transformation, e.g. with [23]. Another option is to use an implementation for projective equivalence called `ProjectiveHom`, which finds both r and the gauge transformation [24].*

Note that to prove that the algorithm will succeed when L is absolutely irreducible and L_6 has two right-factors of order 3, we need to prove that the operator L_s from Step 4 is projectively equivalent to L, i.e., we need to prove that the projective equivalence that Step 5 aims to compute actually exists. This is proved in Proposition 3.7.9.

Algorithm: `Case 3(b)`
Input: An irreducible order 4 operator L that is not gauge equivalent to a symmetric square.
Output: Same as Case 3(a) except with L replaced with $L^{(2)}$.

Inefficient version:

1. Simply apply algorithm Case 3(a) to $L^{(2)}$.

Efficient version:

1. Let $L_6 := \bigwedge^2(L)$.
2. Use the efficiency improvement in Section 4.3 to compute order 3 right-factors L_{3a}, L_{3b} of $L_6^{(2)}$ (FAIL if not found).
3. Apply ReduceOrder to L_{3a} to find L_{2a} (FAIL if not found).
   Substitute $x \mapsto x + 1/2$ in L_{2a} to find L_{2b}.
   (The proof of Theorem 4.3.1 shows that this substitution swaps L_{3a} and L_{3b} up to gauge-equivalence).

4. Now proceed as in Case 3(a).

6.1.1 Example: OEIS A227845

Let

$$a(n) = \sum_{k=0}^{[n/2]} \sum_{j=k}^{n-k} \binom{n-k}{j}^2 \binom{j}{k}^2$$

with $L = (x+4)^2\Phi^4 - 2(3x^2+21x+37)\Phi^3 + 2(3x^2+15x+19)\Phi - (x+2)^2$.

With algorithm Case 3(b) we found that $L^{(2)}$ is not just gauge equivalent, but actually equal to $L_{2a} \circledS L_{2b}$ for some L_{2a} and L_{2b} where L_{2b} is a $x \mapsto x + 1/2$ shift from L_{2a}. We can write $a(2n)$ as a product of their solutions.

As for $a(n)$ itself, we listed the formula $a(n) = U(n)\cdot U(n-1)$ on the OEIS, where $U(-1), U(1), U(3), \ldots$ and $U(0), U(2), U(4), \ldots$ satisfy the recurrence $n^2U(n) = 2(3n^2-3n+1)U(n-2) - (n-1)^2U(n-4)$ obtained by applying the isomorphism $\psi_2 : D \to D_2$ to L_{2b}.

The examples here give only summaries of the computations (omitting large expressions such as gauge transformations). Detailed computations are provided online [24, ReduceOrder.mw].

6.1.2 Example: OEIS A247365

Let $L = (16x^6+96x^5+237x^4+307x^3+222x^2+88x+15)\Phi^4 + (x+1)(64x^8+800x^7+4244x^6+12430x^5+21920x^4+23837x^3+15726x^2+5872x+972)\Phi^3 + (-256x^{10}-3840x^9-25472x^8-98304x^7-244271x^6-408233x^5-464965x^4-357285x^3-178432x^2-53022x-7290)\Phi^2 + (-64x^9-800x^8-4244x^7-12422x^6-21868x^5-23719x^4-15610x^3-5847x^2-1010x-6)\Phi + 16x^6+192x^5+957x^4+2535x^3+3765x^2+2977x+981$.

Algorithm Case 3(a) finds two operators, projectively equivalent to $\Phi^2 + \Phi x - 1$ and $(2x-1)\Phi^2 + 2(4x^2+1)x\Phi + 2x + 1$, which can be solved in terms of Bessel sequences using the algorithm from [9]. This led to the following formula

$$a(n) = [2 \cdot \text{BesselI}(n-1, 2) \cdot \text{BesselK}(2n-1, 2)]$$

where [ ] rounds to the nearest integer.

6.2 Case (4): Symmetric Cube

If an irreducible fourth order operator L is projective equivalent to a symmetric cube, then like in [38], either $L^{\circledS 2}$ has order 7, or it has order 10 with factors of orders 3 and 7. The implementation is at [24].

Algorithm: `Case 4`

Input: An irreducible order 4 operator L.

Output: FAIL if L is not projectively equivalent to $L_2^{\circledS 3}$ for some L_2 of order 2, else L_2 and a projective transformation.

1. Find $L^{\circledS 2}$.
2. If $\text{order}(L^{\circledS 2}) = 7$, use algorithm `SpecialCase` below.
3. Let $L_3 = \text{RightFactors}(L^{\circledS 2}, 3)$. If no right factor was found then return FAIL.
4. Apply ReduceOrder to L_3 obtain an order 2 operator L_2.
5. $L_s := L_2^{\circledS 3}$.
6. Find a projective transformation like in Remark 6.1.1.

The factorization in Step 3 in algorithm `Case 4` typically involves a huge number of candidate types, which makes the algorithm very slow. A large speedup is obtained if we tell RightFactors which candidates types to discard with the following lemma. We implemented [24] an additional speedup by discarding candidate Newton polygon/polynomials that are incompatible with projective equivalence to a symmetric square.

Lemma 6.2.1. *Any candidate type in the factorization in Step 3 in algorithm `Case 4` whose square is not* $\sim \det(L)^3$ *can be dicarded.*

Proof. Given L, the algorithm decides if $D/DL \cong \text{Sym}^3(M_2) \otimes M_1$ for some modules M_1 and M_2 of dimension 1 and 2. Let $d_1 := \det(M_1)$, $d_2 := \det(M_2)$, and $M := \text{Sym}^3(M_2) \otimes M_1$. Then $\det(L) \sim \det(M) = d_1^4 d_2^6$, see Lemma 2.2.1. So if $\det(L)$ is not $\sim$ to a square, then the algorithm can

give up. Now $M_{10} := \mathrm{Sym}^2(M)$ ($\cong D/DL^{\mathbb{S}2}$ if this has order 10) has a surjective homomorphism to $M_7 := \mathrm{Sym}^6(M_2) \otimes \mathrm{Sym}^2(M_1)$.

So its kernel, the 3-dimensional module M_3 that `RightFactors` should find, has $\det(M_3) \sim \det(M_{10})/\det(M_7) = \det(M)^5/(d_1^{14}d_2^{21}) = d_1^6 d_2^9$. The square of this is $\sim \det(L)^3$. □

Algorithm: `SpecialCase`

Input: An irreducible order 4 operator for which $L_4^{\mathbb{S}2}$ has order 7.

Output: Same as algorithm `Case 4`.

Explanation: Let $b_1, \ldots, b_4 := \Phi^0, \ldots, \Phi^3$ be the standard basis of $M := D/DL$, then $B_2 := \{b_1^2, b_1b_2, \ldots, b_4^2\}$ from Section 2.1 is a basis of $M_{10} := \mathrm{Sym}^2(M)$, and $L^{\mathbb{S}2}$ from Section 2.2 is the minimal operator of b_1^2. If this has order 7 then $D/DL^{\mathbb{S}2} \cong M_7$ with M_7 as above. To obtain L_3 we have to compute a cyclic vector of M_{10}/M_7. The computation of $L^{\mathbb{S}2}$ already involved computing a basis of M_7.

1. Pick any $b \in B_2$ with $b \notin M_7$.
2. Compute $b, \phi(b), \phi^2(b), \phi^3(b) \in M_{10}$.
3. Solve linear equations to find L_3 with $L_3(b) \equiv 0 \bmod M_7$.
4. Proceed as in algorithm `Case 4`.

6.2.1 Example: OEIS A219670

Let $L = (x+3)^2(x+4)^2(x+5)(2x+3)(7x^4+56x^3+166x^2+216x+105)\Phi^4-(x+3)(x+4)(2x+3)(2x+7)(70x^6+1050x^5+6406x^4+20337x^3+35449x^2+32244x+12048)\Phi^3-3(x+3)(2x+5)(490x^8+9800x^7+84910x^6+416150x^5+1261159x^4+2417840x^3+2860095x^2+1905600x+546588)\Phi^2+27(x+2)^2(2x+3)(2x+7)(70x^6+1050x^5+6406x^4+20283x^3+35044x^2+31221x+11178)\Phi+729(x+1)^3(x+2)^2(2x+7)(7x^4+84x^3+376x^2+744x+550)$.

Our algorithm finds that L is projectively equivalent to the symmetric cube of $\Phi^2+(2x+1)\Phi-3x^2$, based on the worksheet in [24]. After solving this in terms of OEIS sequences with [35] and some simplification, we obtained the following formula

$$a(n) = A002426(n)^2 A005717(n).$$

(Note: A005717 is gauge equivalent to A002426; L is projective equivalent but not equal to a symmetric cube).

For more details on these examples, such as timings, the number of combinations the factorizer checks (before or after the efficiency improvements in this paper), or other examples, see [24].

APPENDIX A

MORE EXPLANATION ON PROPOSITION 3.2.1

We want to give an explanation about why the relations in (3.1) and $\det(g_{ij}) \neq 0$ forces D to be diagonal or skew-diagonal:

Suppose all constraints fail. In other words $g_{11}g_{21} = 0, g_{21}g_{31} = 0, g_{13}g_{23} = 0$, and $g_{23}g_{33} = 0$. We could have $g_{11} = 0$ or $g_{11} \neq 0, g_{21} = 0$, and $g_{13} = 0$ or $g_{13} \neq 0$, and $g_{23} = 0$. Thus either $g_{21} = 0$ or $g_{11} = g_{31} = 0$, and either $g_{23} = 0$ or $g_{13} = g_{33} = 0$.

Now, if $g_{11} = g_{31} = 0$, or $g_{13} = g_{33} = 0$ then we get a full column with zeroes. So, these two cases are not possible.

This reduces to the case where $g_{21} = 0$ and $g_{23} = 0$. Using 3.2, this means that $a_{11} = 0$ or $a_{21} = 0$, and $a_{12} = 0$ or $a_{22} = 0$.

So, we have 4 different cases to consider now:

1. $a_{11} = 0$ and $a_{12} = 0$, making the first row in 3.2 all zeroes, which is not possible.
2. $a_{21} = 0$ and $a_{22} = 0$, making the third row all zeroes, which is not possible.
3. $a_{11} = 0$ and $a_{22} = 0$, making the matrix skew diagonal.
4. $a_{21} = 0$ and $a_{12} = 0$, making the matrix diagonal.

APPENDIX B

A GRÖBNER BASIS COMPUTATION

This appendix is reproduced from [7].

We now verify a computation needed in the proof of Proposition 3.2.1. Namely, we show the relations in (3.2) are contained in the ideal generated by the relations (3.1) and $1 - \det(g_{ij}) \cdot z$ (the latter relation is used to ensure that $\det(g_{ij})$ is nonzero by the Rabinowitsch trick). This verification is done using Gröbner bases in the next three steps and the Python code that follows.

1. First, we compute a Gröbner basis `I_groebner` of the ideal `I` generated by the relations (3.1) and $1 - \det(g_{ij}) \cdot z$.
2. Second, we augment this Gröbner basis by the scaled relations `J` coming from (3.2). We call the resulting list `IJ`. In more details, we let `G` and `A` store the entries on the left and right sides of (3.2), respectively. We let `J` store the relations (3.2) after scaling by $4g_{11}g_{21}^2$ to clear denominators. The list `IJ` concatenates the lists `J` and `I_groebner`.
3. Finally, we see that the Gröbner basis `IJ_groebner` of the augmented list `IJ` is identical to the Gröbner basis `I_groebner` of `I` since line `17` returns `True`. Therefore the relations in `J` are already generated by the relations in `I`.

```
import sympy
from sympy import symbols, groebner
from sympy.matrices import Matrix

a11, a12, a21, a22, g11, g12, g13, g21, g22, g23, g31, g32, g33, z = symbols('a11 a12 a21 a22 g11 g12 g13
↪  g21 g22 g23 g31 g32 g33 z')

G = Matrix([[g11, g12, g13], [g21, g22, g23], [g31, g32, g33]])
A = g11 * Matrix([[a11**2, 2*a11*a12, a12**2], [a11*a21, a11*a22+a12*a21, a12*a22], [a21**2, 2*a21*a22,
↪  a22**2]])

I = [1-G.det()*z, g21**2-g11*g31, g23**2-g13*g33, 2*g21*g22-g11*g32-g31*g12, 2*g22*g23-g12*g33-g32*g13,
↪  g22**2-g12*g32 + 2*g21*g23-g11*g33-g31*g13]
I_groebner = groebner(I, g11, g12, g13, g21, g22, g23, g31, g32, g33, z)

J = (G - A.subs([(a11, 1), (a12, g12/(2*g11)), (a21, g21/g11), (a22, g32/(2*g21))])) * 4 * g11 * g21**2
IJ = J[:][:] + I_groebner[:]
IJ_groebner = groebner(IJ, g11, g12, g13, g21, g22, g23, g31, g32, g33, z)

IJ_groebner == I_groebner # returns True
```

APPENDIX C

MACKEY'S FORMULA

This appendix is reproduced from [7].

In this appendix, we refine the decomposition in Proposition 3.4.2 in analogy to representation theory.

Theorem C.0.1 (Mackey's formula)**.** *Let $s, t \geq 1$, $d = \gcd(s,t)$, and $l = \operatorname{lcm}(s,t)$. Given a D_s-module N, we have a D_t-module isomorphism*

$$N{\uparrow_s^1}{\downarrow_t^1} \cong \bigoplus_{i=0}^{d-1} \left(\Phi^i \otimes N\right) {\downarrow_l^s}{\uparrow_l^t}. \tag{C.1}$$

Proof. Consider the isomorphisms

$$D \cong \bigoplus_{i=0}^{d-1} D_d \otimes \Phi^i \cong \bigoplus_{i=0}^{d-1} D_t \otimes_{D_l} (\Phi^i \otimes D_s)$$

where the second isomorphism sends $\Phi^{jt+ks} \otimes \Phi^i$ to $\Phi^{jt} \otimes \Phi^i \otimes \Phi^{ks}$. The first and third term are isomorphic as (D_t, D_s)-bimodules. Therefore

$$\begin{aligned} N{\uparrow_s^1}{\downarrow_t^1} &\cong (D \otimes_{D_s} N){\downarrow_t^1} \cong \bigoplus_{i=0}^{d-1} D_t \otimes_{D_l} \left(\Phi^i \otimes D_s\right) \otimes_{D_s} N \\ &\cong \bigoplus_{i=0}^{d-1} D_t \otimes_{D_l} \left(\Phi^i \otimes N\right) {\downarrow_l^s} \cong \bigoplus_{i=0}^{d-1} \left(\Phi^i \otimes N\right) {\downarrow_l^s}{\uparrow_l^t}. \end{aligned}$$

□

Proposition C.0.2. *If N is absolutely irreducible, then the direct summands $(\Phi^i \otimes N){\downarrow_l^s}{\uparrow_l^t}$ in* (C.1) *are irreducible and induced from the absolutely irreducible modules $(\Phi^i \otimes N){\downarrow_l^s}$.*

Proof. Since N is absolutely irreducible, so is $\Phi^i \otimes N$, and by the definition of absolute irreducibility, $(\Phi^i \otimes N){\downarrow_l^s}$ as well.

To see that $\tilde{N} \coloneqq (\Phi^i \otimes N)\downarrow_l^s \uparrow_l^t$ is irreducible, let P be a nonzero irreducible D_t-submodule of $\tilde{N}$. We must show that $P = \tilde{N}$. To do this, we first consider the restriction $P\downarrow_l^t$, which is a D_l-submodule of the restriction $\tilde{N}\downarrow_l^t$. Using (3.5) of Remark 3.3.2 we can decompose $\tilde{N}\downarrow_l^t$ as

$$\tilde{N}\downarrow_l^t = (\Phi^i \otimes N)\downarrow_l^s \uparrow_l^t \downarrow_l^t = (\Phi^i \otimes N)\downarrow_l^s \oplus (\Phi^{i+t} \otimes N)\downarrow_l^s \\ \oplus (\Phi^{i+2t} \otimes N)\downarrow_l^s \oplus \cdots \oplus (\Phi^{i+(l-t)} \otimes N)\downarrow_l^s \quad \text{(C.2)}$$

Since we just proved that each such direct summand in (C.2) is absolutely irreducible (hence irreducible), at least one of these direct summands must lie within $P\downarrow_l^t$. But $P\downarrow_l^t$ is just P viewed as a D_l-module, so P will contain the same direct summand and additionally all its Φ^t-translates. But Φ^t acts transitively on the direct summands of (C.2), so P contains all the direct summands of (C.2). In other words, $P = \tilde{N}$, and so $\tilde{N}$ is irreducible. $\square$

APPENDIX D

MORE RESULTS ON ABSOLUTE IRREDUCIBILITY

This appendix is reproduced from [7].

The following theorem clarifies why our definition of absolute irreducibility agrees with other common usages of the term.

Theorem D.0.1. *For $L \in D$ and $G = \mathrm{Gal}(L)$, the following are equivalent.*

1. *$V(L)$ is an irreducible H-module for every algebraic subgroup H of G with $[G : H]$ finite.*
2. *$V(L)$ is an irreducible G°-module, where G° is the connected component of the identity in G.*
3. *L is absolutely irreducible.*

Proof. We may assume that L is irreducible, otherwise (1)–(3) are all false. (By Proposition 2.3.2, $V(L)$ is a reducible G-module iff L is reducible over $\mathbb{C}(x)$.)

(1)$\Leftrightarrow$(2): Clearly (2) is the special case $H = G^\circ$ of (1). Conversely, let H be a finite-index, closed subgroup of G. By [47, Proposition 2.2.1], H contains G°. So if $V(L)$ is irreducible as a G°-module, then it is also irreducible as an H-module.

(1)$\Rightarrow$(3): We will prove the contrapositive of (1)$\Rightarrow$(3). Suppose that L is not absolutely irreducible. By Theorem 3.4.4, $M_L \cong N{\uparrow}_s^1$ for some $s > 1$. By the discussion after Remark 3.3.3, L is gauge equivalent to some $L' \in D_s$.

Since replacing L by L' does not alter the truth of the implication (1)$\Rightarrow$(3), we can assume that L is in D_s. So now L only contains powers of Φ^s, and we can write

$$V(L) = V(L)e_0 \oplus \cdots \oplus V(L)e_{s-1}$$

for some orthogonal idempotents e_i within the Picard-Vessiot ring R of $\mathrm{LCLM}(L, \Phi^s - 1)$, with $(\Phi^s - 1)e_i = 0$ for $i = 0, \ldots, s-1$.

We first argue that $\mathrm{Gal}(R)$ permutes $\{e_0, \ldots, e_{s-1}\}$. Since $\Phi^s - 1$ has coefficients over $\mathbb{C}(x)$, $\mathrm{Gal}(R)$ leaves $V(\Phi^s - 1)$ invariant. Since $\mathrm{Gal}(R)$ acts as difference ring homomorphisms on R, an orthogonal set of s-many idempotents within $V(\Phi^s - 1)$ must be sent to another orthogonal set of s-many idempotents within $V(\Phi^s - 1)$. But the only such set is $\{e_0, \ldots, e_{s-1}\}$, and so $\mathrm{Gal}(R)$ permutes it.

Thus $\mathrm{Gal}(R)$ also acts by permutation on $\{V(L)e_0, \ldots, V(L)e_{s-1}\}$, and the stabilizer K of $V(L)e_0$ is a finite-index subgroup of $\mathrm{Gal}(R)$. Consequently, $K \cap G$ is a finite-index subgroup of G that continues to stabilize $V(L)e_0$. Therefore (1) fails by taking $H = K \cap G$.

(3)⇒(2): Let $M = D/DL$. By Proposition 3.3.5 (2), there exists t for which $\mathrm{Gal}(M{\downarrow_t^1}) = G^\circ$. If $V(L)$ is a reducible G°-module, then the same is true for $V(M{\downarrow_t^1})$. Then $M{\downarrow_t^1}$ is reducible, and M is not absolutely irreducible. □

APPENDIX E

TENSOR PRODUCT OF TWO SEMISIMPLE MODULES

This appendix is [42, Exercise 2.38 (4)].

Remark E.0.1. *When G is defined over an algebraically closed field of characteristic zero, it is known that G is reductive if and only if it has a faithful completely reducible G-module. In this case, all G-modules will be completely reducible.*

Theorem E.0.2. *[46, Lemma 2.13] Let M be a differential module. M is completely reducible iff its differential Galois group is reductive.*

Theorem E.0.3. *Let M be completely reducible differential module. Every object N of $\{\{M\}\}$ is completely reducible.*

Proof. Using Tannaka duality, the differential module M will give rise to a representation of algebraic group G. By Theorem E.0.2, G is reductive and so every representation of G is completely reducible, by Remark E.0.1.

Tannakian category generated by M consists of all differential modules obtained from M by taking direct sums, tensor products, duals, sub-modules, and quotient modules. Thus, every N of $\{\{M\}\}$ should be completely reducible. □

Theorem E.0.4. *The tensor product $M_1 \otimes M_2$ of two completely reducible modules is again completely reducible.*

Proof. Let $M = M_1 \oplus M_2$. M is a direct sum of completely reducible modules, therefore, completely reducible. We have that $M_1 \otimes M_2$ is a sub-module of $M \otimes M$. By Theorem E.0.3, $M_1 \otimes M_2$ is completely reducible. □

APPENDIX F

NON EULERIAN GROUPS

Table 3.1 mentioned that $\mathrm{Sp}_4(\mathbb{C})$ and $\mathrm{SL}_4(\mathbb{C})$ are not Eulerian. Although this claim can be proved directly, in this appendix we list results from the differential case that imply this claim. For basic introduction to lie algebras and groups see [29].

Theorem F.0.1. *([45, Theorem 1]) Let $L(y) = 0$ be a homogeneous linear differential equation of order n, $n \geq 3$ with coefficients in $\mathbb{C}(x)$ and let K be the associated Picard-Vessiot extension. Let $g(K/\mathbb{C}(x))$ be the lie algebra of group $G(K/\mathbb{C}(x))$. Assume that $G(K/\mathbb{C}(x)) \subset SL(n, \mathbb{C})$. $L(y) = 0$ can be solved in terms of equations of lower order if and only if one of the following holds:*

(a) $g(K/\mathbb{C}(x)) \subset sl(n, \mathbb{C})$ leaves a nontrivial subspace of $\mathbb{C}^n$ invariant.

(b) $g(K/\mathbb{C}(x))$ is semisimple but not simple.

(c) $g(K/\mathbb{C}(x))$ is simple and there exists a nonzero Lie algebra homomorphism

$$\rho : g(K/\mathbb{C}(x)) \to \mathfrak{gl}(m, \mathbb{C}) \quad \text{for some } m < n.$$

This should also be true in the difference case (M. van der Put, e-mail correspondence).

Corollary F.0.2. *([45, Corollary p.119]) $L(y) = 0$ cannot be solved in terms of equations of lower order if and only if $g(K/\mathbb{C}(x))$ is simple and has no nonzero representation of degree less than n.*

This Corollary states that $g(K/\mathbb{C}(x))$ is a simple Lie algebra that has no nontrivial representation of lower degree. Therefore we must examine those simple Lie algebras g whose lowest dimensional nontrivial representation is of dimension 4. Axelle Person in her thesis [38, p.33], stated that $sl_4(\mathbb{C})$ and $sp_4(\mathbb{C})$ are simple lie algebras of sl_4. We use the fact that $SL(n, \mathbb{C})$ and $Sp(n, \mathbb{C})$ act transitively on $\mathbb{P}(\mathbb{C}^n)$ for $3 \leq n \leq 6$ [27, p. 360 and p. 374]. The transitive action of $SL(n, \mathbb{C})$ or $Sp(n, \mathbb{C})$ on $\mathbb{P}(\mathbb{C}^n)$ implies that the standard representation is irreducible and minimal, so there can be no nontrivial irreducible representation of smaller degree than n. Thus, $\mathrm{Sp}_4(\mathbb{C})$ and $\mathrm{SL}_4(\mathbb{C})$ are non Eulerian.

APPENDIX G

EXTERIOR SQUARE OF A TENSOR PRODUCT

This appendix is reproduced from [8].

If an irreducible operator L of order 4 is gauge equivalent to a symmetric product of second order operators, then $\bigwedge^2(L)$ has two right-factors of order three, both 2-solvable

Let M, N be D-modules with bases $\{m_1, m_2\}$ and $\{n_1, n_2\}$. Then $M \otimes N$ has this basis

$$w_1 = m_1 \otimes n_1, \quad w_2 = m_1 \otimes n_2, \quad w_3 = m_2 \otimes n_1, \quad w_4 = m_2 \otimes n_2$$

and $\wedge^2(M \otimes N)$ has basis $e_1, \ldots, e_6 =$

$$w_1 \wedge w_2,\ w_1 \wedge w_3,\ w_1 \wedge w_4,\ w_2 \wedge w_3,\ w_2 \wedge w_4,\ w_3 \wedge w_4$$

Let $d_m = m_1 \wedge m_2$ resp. $d_n = n_1 \wedge n_2$ be a basis of $\wedge^2 M$ resp. $\wedge^2 N$.

A classical decomposition gives

$$\wedge^2(M \otimes N) \ \cong\ (\mathrm{Sym}^2 M \otimes \wedge^2 N) \ \oplus\ (\wedge^2 M \otimes \mathrm{Sym}^2 N).$$

The isomorphism is given by

$$e_1 \mapsto m_1^2 \otimes d_n, \quad (e_3 - e_4)/2 \mapsto m_1 m_2 \otimes d_n, \quad e_6 \mapsto m_2^2 \otimes d_n$$

$$e_2 \mapsto d_m \otimes n_1^2, \quad (e_3 + e_4)/2 \mapsto d_m \otimes n_1 n_2, \quad e_5 \mapsto d_m \otimes n_2^2.$$

To verify this, observe that ϕ acts on $\{e_1, (e_3 - e_4)/2, e_6, e_2, (e_3 + e_4)/2, e_5\}$ as it acts on $\{m_1^2 \otimes d_n, \ldots, d_m \otimes n_2^2\}$, this action is given by

$$\begin{pmatrix} \det(B)\rho_3(A) & 0_{3\times 3} \\ 0_{3\times 3} & \det(A)\rho_3(B) \end{pmatrix}.$$

Here A, B express the action of ϕ on $\{m_1, m_2\}$ and $\{n_1, n_2\}$, while $\rho_3(A), \rho_3(B)$ express the action on $\{m_1^2, m_1 m_2, m_2^2\}$ and $\{n_1^2, n_1 n_2, n_2^2\}$. The 3×3 matrix $\rho_3(A)$ is also found in the proof of [44, Lemma 3.2].

BIBLIOGRAPHY

[1] B. Adamczewski, T. Dreyfus, and C. Hardouin. Hypertranscendence and linear difference equations. *Journal of the American Mathematical Society*, 34(2):475–503, 2021.

[2] S. Aldossari. Algorithms for simplifying differential equations. PhD thesis, Florida State University, 2020.

[3] K. Amano. Relative invariants, difference equations, and the Picard–Vessiot theory. arXiv:math/0503291, 2005.

[4] M. Barkatou, M. van Hoeij, J. Middeke, and Y. Zhou. Hypergeometric solutions of linear difference systems. *Journal of Symbolic Computation,* 102475, 2025.

[5] S. Bawazeer. Simplifying difference equations. PhD thesis, Florida State University, 2025.

[6] H. Bou KaedBey, M. van Hoeij, and M. C. Tsui. Solving third order linear difference equations in terms of second order equations. In *Proceedings of ISSAC 2024*, pages 457–463, 2024.

[7] H. Bou KaedBey, M. van Hoeij, and M. C. Tsui. Solving order 3 difference equations. *Journal of Symbolic Computation*, 129:102419, 2025.

[8] H. Bou KaedBey and M. van Hoeij. Algorithms for 2-solvable difference equations. *Journal of Symbolic Computation*, 135:102550, 2026.

[9] Y. Cha, M. van Hoeij, and G. Levy. Solving recurrence relations using local invariants. In *Proceedings of ISSAC 2010*, pages 303–309, 2010.

[10] Y. Cha. Closed form solutions of linear difference equations. PhD thesis, Florida State University, 2011.

[11] Y. Cha and M. van Hoeij. Liouvillian solutions of irreducible linear difference equations. In *Proceedings of ISSAC 2009*, pages 87–94, 2009.

[12] R. Debeerst, M. van Hoeij, and W. Koepf. Solving differential equations in terms of Bessel functions. In *Proceedings of ISSAC 2008*, pages 39–46, 2008.

[13] T. Fang and M. van Hoeij. 2-descent for second order linear differential equations. In *Proceedings of ISSAC 2011*, pages 107–114, 2011.

[14] P. A. Hendricks and M. F. Singer. Solving difference equations in finite terms. *Journal of Symbolic Computation*, 27(3):239–259, 1999.

[15] P. A. Hendriks. An algorithm determining the difference Galois group of second order linear difference equations. *Journal of Symbolic Computation*, 26(4):445–462, 1998.

[16] S. A. Hessinger. Computing the Galois group of a linear differential equation of order four. *Applicable Algebra in Engineering, Communication and Computing*, 11(6):489–536, 2001.

[17] M. van Hoeij. Factorization of differential operators with rational function coefficients. *Journal of Symbolic Computation*, 24(5):537–561, 1997.

[18] M. van Hoeij. Decomposing a fourth order linear differential equation as a symmetric product. *Banach Center Publications*, 58:89–96, 2002.

[19] M. van Hoeij and J. Cremona. Solving conics over function fields. *Journal de Théorie des Nombres de Bordeaux*, 18(3):595–606, 2006.

[20] M. van Hoeij. Solving third order linear differential equations in terms of second order equations. In *Proceedings of ISSAC 2007*, pages 355–360, 2007.

[21] M. van Hoeij and Q. Yuan. Finding all Bessel type solutions for linear differential equations with rational function coefficients. In *Proceedings of ISSAC 2010*, pages 37–44, 2010.

[22] M. van Hoeij and M. van der Put. Descent for differential modules and skew fields. *Journal of Algebra*, 296(1):18–55, 2006.

[23] M. van Hoeij and Y. Cha. Implementation for Hom. `https://www.math.fsu.edu/~hoeij/papers/Hom/index.html`.

[24] M. van Hoeij and H. Bou KaedBey. Implementation and examples. `https://www.math.fsu.edu/~hboukaed/Implementations`, `https://www.math.fsu.edu/~hoeij/AbsFactor`. Newer version `https://www.math.fsu.edu/~hoeij/LRE`

[25] M. van Hoeij. Implementations for RFactors. `https://www.math.fsu.edu/~hoeij/algorithms/RFactors/`.

[26] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory*. Springer, 2012.

[27] N. Jacobson. *Basic Algebra I*. Dover, 2012.

[28] I. Kaplansky. *An Introduction to Differential Algebra*. Hermann, Paris, 1957.

[29] A. A. Kirillov. *An Introduction to Lie Groups and Lie Algebras*. Cambridge University Press, 2008.

[30] J. J. Kovacic. An algorithm for solving second order linear homogeneous differential equations. *Journal of Symbolic Computation*, 2(1):3–43, 1986.

[31] T. Y. Lam. *Introduction to Quadratic Forms over Fields*. American Mathematical Society, 2005.

[32] S. Lang. *Introduction to Transcendental Numbers*. Addison-Wesley, 1966.

[33] S. Lang. *Cyclotomic Fields I and II*. Springer, 2012.

[34] C. Lassueur and N. Farrell. *Modular Representation Theory of Finite Groups*. Cambridge University Press, 2020.

[35] G. Levy. Solutions of second order recurrence relations. PhD thesis, Florida State University, 2010.

[36] A. K. Nguyen. On d-solvability for linear differential equations. *Journal of Symbolic Computation*, 44(5):421–434, 2009.

[37] K. A. Nguyen and M. van der Put. Solving differential equations. *Pure and Applied Mathematics Quarterly* 6.1: 173-208, 2010.

[38] A. C. Person. Solving homogeneous linear differential equations of order four in terms of equations of smaller order. PhD thesis, North Carolina State University, 2002.

[39] M. Petkovšek. Hypergeometric solutions of linear recurrences with polynomial coefficients. *Journal of Symbolic Computation*, 11:1–22, 1992.

[40] M. Petkovšek, H. S. Wilf, and D. Zeilberger. *A=B*. A K Peters, 1997.

[41] M. van der Put and M. F. Singer. *Galois Theory of Difference Equations*. Springer, 2006.

[42] M. van der Put and M. F. Singer. *Galois Theory of Linear Differential Equations*. Springer, 2012.

[43] J.-P. Serre. *Linear Representations of Finite Groups*. Springer, 1977.

[44] M. F. Singer. Solving homogeneous linear differential equations in terms of second order linear differential equations. *American Journal of Mathematics*, 107(3):663–696, 1985.

[45] M. F. Singer. Algebraic relations among solutions of linear differential equations: Fano's theorem. *American Journal of Mathematics*, 110(1):115–143, 1988.

[46] M. F. Singer. Testing reducibility of linear differential operators: a group theoretic perspective. *Applicable Algebra in Engineering, Communication and Computing*, 7(2):77–104, 1996.

[47] T. A. Springer. *Linear Algebraic Groups*. Springer, 1998.

[48] Z. W. Sun. On Motzkin numbers and central trinomial coefficients. *Advances in Applied Mathematics,* 136: 102319, 2022.

[49] G. Zehfuss. Ueber eine gewisse Determinante. *Zeitschrift für Mathematik und Physik*, 3:298–301, 1858.

[50] Y. Zhou. Algorithms for factoring linear recurrence operators. PhD thesis, Florida State University, 2022.

BIOGRAPHICAL SKETCH

EDUCATION

Ph.D in Mathematics, May 2026, Florida State University.

Advisor: Professor Mark Van Hoeij

MS in Mathematics, July 2022, Florida State University.

MS in Mathematics, May 2019, Notre Dame University-Lebanon.

Advisor: Professor Jessica Merhej

BS in Mathematics, May 2017, Notre Dame University-Lebanon.

HONORS AND AWARDS

Erdős Institute Deep Learning Certificate, November 2025

Erdős Institute Data Science Certificate, December 2024 and April 2025

Distinguished Teaching Award, April 2025

Clara Kibler Davis Award, April 2021, 2023, 2024 and 2025 (Outstanding Women in STEM)

Tam Family Award, April 2024 (Conference Travel Grant)

Bettye Anne Busbee Graduate Fellowship, April 2023 (Recognizing Outstanding Scholarly Work)

EXPERIENCE

Instructor of Record, August 2023 - May 2026, Florida State University.

- Ordinary Differential Equations, Spring'26, an undergraduate course averaging 29 students, covering the following topics: differential equations of the first order, linear equations of the second order, power series solutions, and Laplace transforms.

- Calculus II, Summer'24 - Fall'24 - Spring'25, an undergraduate course averaging 19 students per semester, covering the following topics: integrals, sequences and series and differential equations.
- Calculus I, Fall'23 - Spring'24, an undergraduate course averaging 29 students per semester, covering the following topics: limits, derivatives and integrals.
- Created quizzes, exams and homework.
- Prepared notes that helped the students study for their exams.
- Implemented poster sessions in Calculus I to enhance interactivity and provide students with the chance to present and explain their work to their peers.

Instructor of Record, September 2017-May 2019, Notre Dame University-Lebanon.

- Mathematics for Business and Economics II, Spring'18, an undergraduate course averaging 19 students per semester, covering the following topics: Accumulation and discounting, simple and compound interest, effective and nominal interest, discount rates, forces of interest and discount, varying interest. Equations of value.
- Calculus and Analytic Geometry I, Fall'17 - Spring'18-Fall'18 - Spring'19, an undergraduate course averaging 30 students per semester, covering the following topics: limits, derivatives and integrals.
- Principle of Calculus, Fall'17 - Fall'18 - Spring'19, an undergraduate course averaging 30 students per semester, covering the following topics: polynomials, functions, systems of equations and inequalities.
- Prepared and presented lessons that enabled students to learn the mathematics skills and concepts included in their curriculum.
- Developed and maintained a comfortable and orderly classroom that promotes and supports learning.

Research Assistant, August 2022-August 2023, Florida State University.

- Classified order three difference operators over $\mathbb{C}(x)$ that are expressible in terms of second order difference operators, and using Galois theory to prove that the list of cases is complete.
- Gave an algorithm for absolute factorization of recurrence operators and an algorithm to solve third order difference equations in terms of second order, similar to existing algorithm for differential equations.

Teaching Assistant, August 2020-July 2022, Florida State University.

- Recitation Instructor for Calculus I, Fall'21
- Lab/Lecture Teaching Assistant for:
  - MAC 2233-Business Calculus, Fall'20 – Spring'21 – Spring'22 - Summer'25
  - MAC 1114-Analytical Trigonometry, Summer'21
  - MAC 1105-College Algebra, Spring'22 - Fall'25
  - MGF 1106-Mathematics for Liberal Arts, Fall'20 – Spring'21
  - MAC 1140-Precalculus, Summer'22

Data Science Teaching Assistant, May 2025-June 2025, Erdős Data Science Bootcamp

- Selected as a TA after passing a rigorous assessment based on data science and machine learning fundamentals.
- Assisted in twice-weekly problem sessions, guiding trainees through applied data science exercises, which improved their understanding and application of data science concepts.

Deep Learning Trainee, September 2025 - May 2026, Erdős Deep Learning Bootcamp

- Learned core deep learning foundations through lectures and self-study using FastAI and PyTorch books, covering fundamentals, binary and multi-class classification, tabular modeling, and recommender systems.

- Applied these methods in a comprehensive real-world project detecting suspicious financial transactions.

Data Science Trainee, September 2024-June 2025, Erdős Data Science Bootcamp

- Strengthened Python and machine learning expertise through structured lectures and hands-on projects.
- Developed and deployed ML applications using real-world datasets with Hugging Face tools.

Private Tutor, May 2019-July 2020, Lebanon.

- Prepared lesson plans on complex numbers, functions, system of equations, inequalities, matrix algebra, conics, sequences, series and induction, tailored to individual student needs.
- Provided study tips that improved student's study skills.
- Devised tutoring methods that helped students score increase by 40 percent.

PROFESSIONAL AFFILIATION

- **SIAM Member,** September 2024-May 2026.
- **Mathematics Graduate Student Council Member,** September 2022-December 2024.
- **Graduate Student Seminar Organizer,** September 2022-December 2024
- **Directed Reading Program Co-Chair,** August 2023-December 2023

PUBLICATIONS AND PRESENTATION

- Journal of Symbolic Computation as a special issue for ISSAC 2024 (2026, Published): "Algorithms for 2-Solvable Difference Equations".

- Journal of Symbolic Computation (2025, Published): "Solving Order 3 Difference Equations".
- ISSAC 2024 Conference Paper: "Solving Third Order Linear Difference Equations in Terms of Second Order Equations".
- Conference Presentation: ISSAC 2024, Raleigh, NC

PROJECTS

- **Deep Learning for Anti-Money Laundering: Detecting Suspicious Transactions (Top Project),** September 2025-December 2025, Erdős Deep Learning Bootcamp.
- **Auto Loan Credit Decisioning Model,** January 2025-April 2025, Erdős Data Science Bootcamp.
- **Fake Image Detection Model,** September 2024-December 2024, Erdős Data Science Bootcamp.