Rational Elements of the Tensor Product of Solutions of Difference Operators (Extended Abstract)

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1 Introduction

In [5], an algorithm was given to solve difference equations. One of the main ingredients was computing homomorphisms between solution spaces of difference operators. The algorithm was implemented for order 2. To extend it to higher order, we need to compute homomorphisms for higher order operators. Here we present an algorithm that computes rational (invariant under the difference Galois group) elements of the tensor product of the solution spaces of two difference operators (the main ingredient for computing Hom).

2 Preliminaries

Let τ be a shift operator, $x \mapsto x+1$ and $D := \mathbb{C}(x)[\tau, \tau^{-1}]$ where multiplication is given by: $\tau \cdot f(x) = f(x+1)\tau$. In this paper, we use $L \in D$ of the form $L = a_d(x)\tau^d + \cdots + a_0(x)$ with $a_d(x)a_0(x) \neq 0$, for such L, we denote $\mathrm{ord}(L) := d$. Let V be a universal extension of $\mathbb{C}(x)$, which means $\dim_{\mathbb{C}}(\ker(L, V)) = \mathrm{ord}(L)$ for any $L \neq 0$, see [8] for more details. We denote $V(L) := \ker(L, V)$. We denote $L^{(i)} := \tau^i \cdot L \cdot \tau^{-i}$ so that $V(L^{(i)}) = \tau^i(V(L))$.

3 Rational elements of $V(M) \otimes V(N)$

For $M, N \in D$, we denote the symmetric product of M and N as $M \circledS N$. It is the minimal order operator in D for which $\mu\nu \in V(M \circledS N)$ for all $\mu \in V(M)$

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and $\nu \in V(N)$, [7].

For $M, N \in D$ the natural map

$$
\begin{aligned}
V(M) \otimes V(N) &\to V(M \circledS N) \\
\mu \otimes \nu &\mapsto \mu\nu.
\end{aligned}
$$

is onto, however, in general, it is not injective. Suppose $\mathrm{ord}(M) = m$ and $\mathrm{ord}(N) = n$. To get a space that is isomorphic to $V(M) \otimes V(N)$ we define a map

$$\Phi : V(M) \otimes V(N) \to \mathrm{Mat}_{m,n}(V)$$

by

$$
\Phi(\mu \otimes \nu) := \begin{pmatrix}
\mu\nu & \mu\tau(\nu) & \cdots & \mu\tau^{n-1}(\nu) \\
\tau(\mu)\nu & \tau(\mu)\tau(\nu) & \cdots & \tau(\mu)\tau^{n-1}(\nu) \\
\vdots & \vdots & \ddots & \vdots \\
\tau^{m-1}(\mu)\nu & \tau^{m-1}(\mu)\tau(\nu) & \cdots & \tau^{m-1}(\mu)\tau^{n-1}(\nu)
\end{pmatrix}
$$

Then Φ is injective, and $V(M) \otimes V(N)$ is isomorphic to its image under Φ.

Let $G := \mathrm{Aut}_D(V/\mathbb{C}(x))$ be the difference Galois group, then V, $V(M)$, $V(N)$, $V(M) \otimes V(N)$ and $\mathrm{Mat}_{m,n}(V)$ are G-modules. An element in a G-module is called rational if it is G-invariant. An element of $V(M) \otimes V(N)$ is rational (G-invariant) if and only if its image under Φ is in $\mathrm{Mat}_{m,n}(\mathbb{C}(x))$.

4 Computing rational elements

We say $d(x) \in \mathbb{C}(x)$ is a denominator bound of $f(x)$ if $d(x)f(x) \in \mathbb{C}[x]$, see [1] for details. The paper [9] computes certain sequences $B_i^r(p)$ and $B_i^l(p)$ for each finite singularity $p \in \mathbb{C}/\mathbb{Z}$, in order to construct denominator bounds for rational solutions of an operator M. We compute sequences $B_i^r(p)$ and $B_i^l(p)$ for both M and N in order to construct a denominator-bound for each entry of $\Phi(A)$ where $A \in V(M) \otimes V(N)$. To compute upper bounds for the degrees of the numerators, we multiply generalized exponents of M and N, see [4, 5]. The i,j-th entry of $\Phi(A)$ is a solution of $M^{(i-1)} \circledS N^{(j-1)}$, so what we compute is a bound for the rational solutions of this symmetric product operator (however, we do not explicitly compute $M^{(i-1)} \circledS N^{(j-1)}$, because this expression could be large, and is in any case unnecessary).

We will illustrate our algorithm by an example. Suppose $\mathrm{ord}(M) = 3$ and $\mathrm{ord}(N) = 5$ and we denote $w^{i,j} \in V(M^{(i)} \circledS N^{(j)}) \cap \mathbb{C}(x)$.

$$
\begin{array}{cc|cccccc}
 & & \overset{j}{\longrightarrow} & & & & & \\
 & & & & b_1 & b_2 & b_3 & \\
i & & w^{0,0} & w^{0,1} & w^{0,2} & \tau(b_1) & \tau(b_2) & \tau(b_3) \\
 & \downarrow & & \tau(w^{0,0}) & \tau(w^{0,1}) & \tau(w^{0,2}) & \tau^2(b_1) & \\
 & & & & \tau^2(w^{0,0}) & \tau^2(w^{0,1}) & \tau^2(w^{0,2}) &
\end{array}
$$

First we compute the sequences $B_i^r(q)$ and $B_i^l(q)$ for M and N, and a construct denominator bound $d_j(x)$ for $M \circledS N^{(j)}$, $0 \leq j \leq \mathrm{ord}(M) - 1 = 2$. Then each $w^{0,j}$ can be written as

$$\sum_{k=0}^{n_j} \frac{c_{k,j} x^k}{d_j(x)}$$

for some $n_j \in \mathbb{N}$ and unknowns $c_{k,j}$. The upper bound n_j can be computed by multiplying generalized exponents of M and N. Since $\tau(w^{i,j}) = w^{i+1,j+1}$ we obtain $w^{1,1}$, $w^{1,2}$ and $w^{1,3}$ by applying τ to $w^{0,0}$, $w^{0,1}$ and $w^{0,2}$, respectively and $w^{2,2}$, $w^{2,3}$ and $w^{2,4}$ by applying τ^2 to $w^{0,0}$, $w^{0,1}$ and $w^{0,2}$. Then $b_1 = w^{-1,2}$ can be obtained by applying the recurrence given by M to $w^{0,2}$, $\tau(w^{1,1})$ and $\tau^2(w^{0,0})$. Then we obtain $w^{0,4}$ by applying τ to b_1. Repeating this produces b_2 and b_3. Note that the sequences $B_i^r(q)$ and $B_i^l(q)$ also produce denominator bounds for b_1, b_2, b_3, these bounds can be translated into linear equations for the unknowns $c_{k,j}$ in our ansatz.

After obtaining $w^{0,5} = \tau(b_3)$ by continuing this process, we apply N to $w^{0,0}$, $w^{0,1}$, $w^{0,2}$, $\tau(b_1)$, $\tau(b_2)$ and $\tau(b_3)$, and get a system of equations for $c_{k,j}$. We can add more equations by computing b_4 and b_5, at which point we can prove that the equations are necessary and sufficient. By solving these equations, we obtain a basis of rational elements of $\Phi(V(M) \otimes V(N))$.

The diagram and above process explains the following lemma:

Lemma 1. *If we have* $\min\{\mathrm{ord}(M), \mathrm{ord}(N)\}$ *consecutive, either horizontal or vertical, entries of* $\Phi(A)$ *then all other entries can be computed from them.*

Similar statements are known in other contexts, e.g. computing invariants of the differential Galois group [10].

5 Application

There are different ways to define an adjoint; we choose the following. For $L := \sum_{i=0}^{d} a_i(x)\tau^i \in D, \quad a_d(x) = 1$ we take:

$$L^* := \sum_{i=0}^{d} a_{d-i}(x + i - 1)\tau^i.$$

Then $V(L^*)$ is isomorphic (as G-module) to the dual of $V(L)$, so

$$\mathrm{Hom}_{\mathbb{C}}(V(L_1), V(L_2)) \cong V(L_1)^* \otimes V(L_2) \cong V(L_1^*) \otimes V(L_2).$$

We want to compute

$$\mathrm{Hom}(L_1, L_2) := \{\varphi \in \mathrm{Hom}_{\mathbb{C}}(V(L_1), V(L_2)) \mid \varphi \text{ is } G\text{-invariant}\}.$$

Remark: φ is G-invariant iff $\exists H \in D$ s.t. $\varphi(\nu) = H(\nu)$ for all $\nu \in V(L_1)$. Computing $\mathrm{Hom}(L_1, L_2)$ is equivalent to computing the rational elements of $V(L_1^*) \otimes V(L_2)$.

The paper [2] computes Hom for systems of difference equations, while [3] handles the case of ordinary differential equations and [6] handles the multivariable case. The purpose of this paper is to design a fast algorithm for difference operators, similar to Maple's DEtools[Homomorphisms] for differential operators.

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