Homomorphism between two difference operators

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Notations

- τ is a shift operator, $x \mapsto x+1$
- $D := \mathbb{C}(x)[\tau]$ is a ring of difference operators.

$$\tau \cdot f(x) = f(x+1)\tau, \quad \tau \cdot \tau^i = \tau^{i+1}$$

- $L := \sum_{i=0}^{d} a_i(x)\tau^i \in D$ corresponds to a recurrence equation

$$a_d(x)u(x+d) + \cdots + a_0(x)u(x) = 0$$

  we assume $a_d(x)a_0(x) \neq 0$ and say $\mathrm{ord}(L) = d$
- Let V be the universal extension of D and $V(L) = \ker(L, V)$.
- $L^* := \sum_{i=0}^{d} a_{d-i}(x+i-1)\tau^i$ is called the adjoint of L.

$\mathrm{Hom}(L_1, L_2)$

We want to compute

$$\mathrm{Hom}(L_1, L_2) := \{G \in \mathbb{C}(x)[\tau] \mid \mathrm{ord}(G) < \mathrm{ord}(L_1), G(V(L_1)) \subseteq V(L_2)\}.$$

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Idea:

$$\begin{aligned} \mathrm{Hom}(L_1, L_2) \subseteq\ & \mathrm{Hom}_{\mathbb{C}}(V(L_1), V(L_2)) \\ \cong\ & V(L_1)^* \otimes_{\mathbb{C}} V(L_2) \\ \cong\ & V(L_1^*) \ \otimes_{\mathbb{C}} V(L_2) \end{aligned}$$

Definition

The symmetric product, $N \circledS M$, of N and $M \in D$ is an order-minimal and monic operator such that $\nu\mu \in V(N \circledS M)$ for all $\nu \in V(N)$ and $\mu \in V(M)$.

Suppose $\mathrm{ord}(N) = d_1, \mathrm{ord}(M) = d_2$,

$V(N) := \mathrm{span}_{\mathbb{C}}\{\nu_1, \ldots, \nu_{d_1}\}$ and

$V(M) := \mathrm{span}_{\mathbb{C}}\{\mu_1, \ldots, \mu_{d_2}\}$.

$\Psi : V(N) \otimes V(M) \to V(N \circledS M)$

$\Psi(\sum a_{i,j}\nu_i \otimes \mu_j) = \sum a_{i,j}\nu_i\mu_j$ is an onto map.

$$V(N) \otimes V(M) \ncong V(N \circledS M)$$

For each ν_i and μ_j we define,

$$\mathcal{M}(\nu_i, \mu_j) = \begin{pmatrix} \nu_i \mu_j & \nu_i \tau(\mu_j) & \cdots & \nu_i \tau^{d_2 - 1}(\mu_j) \\ \tau(\nu_i)\mu_j & \tau(\nu_i)\tau(\mu_j) & \cdots & \tau(\nu_i)\tau^{d_2-1}(\mu_j) \\ \vdots & \vdots & \ddots & \vdots \\ \tau^{d_1-1}(\nu_i)\mu_j & \tau^{d_1-1}(\nu_i)\tau(\mu_j) & \cdots & \tau^{d_1-1}(\nu_i)\tau^{d_2-1}(\mu_j) \end{pmatrix},$$

$$Mat(N, M) := \mathrm{span}_{\mathbb{C}}\{\mathcal{M}_{i,j} \mid 1 \le i \le d_1, 1 \le j \le d_2\}.$$

Then

$$V(N) \otimes V(M) \cong Mat(N, M)$$

What is rational in $V(N) \otimes V(M)$?

For $W = \sum a_{i,j}\nu_i \otimes \mu_j \in V(N) \otimes V(M)$ define
$\Phi(W) := \sum a_{i,j}\mathcal{M}(\nu_i, \mu_j) \in Mat(N, M)$.

Definition

$W = \sum a_{i,j}\nu_i \otimes \mu_j \in V(N) \otimes V(M)$ is said to be rational,
if all entries of $\Phi(W)$ are rational.

$$\mathrm{Hom}(L_1, L_2) := \{G \in \mathbb{C}(x)[\tau] \mid \mathrm{ord}(G) < \mathrm{ord}(L_1), G(V(L_1)) \subseteq V(L_2)\}$$

$$\begin{aligned}\mathrm{Hom}(L_1, L_2) &\subseteq \mathrm{Hom}_{\mathbb{C}}(V(L_1), V(L_2)) \\ &\cong V(L_1)^* \otimes_{\mathbb{C}} V(L_2) \\ &\cong V(L_1^*) \otimes_{\mathbb{C}} V(L_2)\end{aligned}$$

Theorem

Rational elements of $V(L_1^*) \otimes_{\mathbb{C}} V(L_2)$ correspond bijectively to elements of $\mathrm{Hom}(L_1, L_2)$.

How to compute rational elements of $Mat(N, M)$

For $W = \sum a_{i,j}\nu_i \otimes \mu_j \in V(N) \otimes V(M)$ define
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$$\Phi(W) = \begin{pmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{pmatrix}$$

$$\Phi(W)_{k,l} = \sum a_{i,j}\tau^{k-1}(\nu_i)\tau^{l-1}(\mu_j)$$

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$\tau(\Phi(W)_{k,l}) = \Phi(W)_{k+1,l+1}$

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$$\begin{array}{cccccc}
 & \vdots & \vdots & & & \\
\cdots & * & * & \cdots & * & \cdots \\
\cdots & * & * & \cdots & * & \cdots \\
 & \vdots & \vdots & \ddots & \vdots & \updownarrow N \\
\cdots & * & * & \cdots & * & \cdots \\
 & \vdots & \vdots & & &
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$$\begin{array}{cccccc}
 & \vdots & \vdots & & \color{red}{*} & \\
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How to compute rational elements of $Mat(N, M)$

We need $\min\{\mathrm{ord}(N), \mathrm{ord}(M)\}$ consecutive elements of $\Phi(W)$ to get all elements in $\Phi(W)$.
Suppose $\mathrm{ord}(N) = 3$ and $\mathrm{ord}(M) = 5$ then

$* \quad * \quad *$

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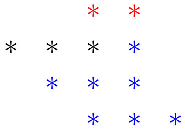

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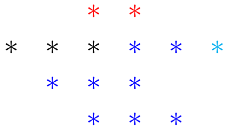

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How to compute rational elements of $Mat(N, M)$

- rational solution
  - $d(x) \in \mathbb{C}(x)$ is a denominator bound of $f(x) \in \mathbb{C}(x)$ if there exists $n(x) \in \mathbb{C}[x]$ such that $f(x) = d(x)n(x)$
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  - degree of $n(x)$ is called the numerator bound of $f(x)$.
  - $d(x)$ can be computed from valuation growth
  - degree of $n(x)$ can be computed from generalized exponents.

How to compute rational elements of $Mat(N, M)$

Let $m \in Mat(N, M)$ be a matrix with rational entries.

- characteristic data
  - a Compute valuation growth of N and M, then we get denominator bound $d_i(n)$ for each $m_{1,i}, i \in \{1, \ldots, d_2 - 1\}$
  - b Compute generalized exponent of M, then we get numerator bound j for each $m_{1,i}, i \in \{1, \ldots, d_2 - 1\}$.
  - c for each $m_{1,i}$ we get $d_i(x) \sum_j c_{i,j} x^j$.
  - d generate m_{1,d_2} with shift and N and apply M to find the coefficients.

Applications of $\mathrm{Hom}(L_1, L_2)$

1. Gauge Transformation

2. Factoring

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   $L_1, L_2 \in D, \mathrm{ord}(L_1) = \mathrm{ord}(L_2)$, are said to be gauge equivalent, $L_1 \sim_g L_2$, if there exist $G \in D$ that bijectively maps $V(L_1) \to V(L_2)$.

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2. Factoring
   For $L \in D$, suppose we can compute $\tilde{M}$ which is gauge equivalent to a right hand factor M of L. Then by applying $G \in \mathrm{Hom}(\tilde{M}, L)$ to $\tilde{M}$, we get M.