

THE FLORIDA STATE UNIVERSITY
COLLEGE OF ARTS AND SCIENCES

CLOSED FORM SOLUTIONS OF LINEAR DIFFERENCE EQUATIONS

By
YONGJAE CHA

A Dissertation submitted to the
Department of Mathematics
in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Degree Awarded:
Spring Semester, 2011

The members of the committee approve the dissertation of Yongjae Cha defended on December 7, 2010.

Mark van Hoeij
Professor Directing Thesis

Robert A. van Engelen
University Representative

Amod Agashe
Committee Member

Ettore Aldrovandi
Committee Member

Paolo Aluffi
Committee Member

Approved:

Philip L. Bowers, Chair, Department of Mathematics

Joseph Travis, Dean, College of Arts and Sciences

The Graduate School has verified and approved the above-named committee members.

To my parents, InHeui Cha and Hangwon Lee,
the love of my life, Soyeon Park, and
my twin sons, Aidan Minjun Cha and Brendan MinSeo Cha.

ACKNOWLEDGMENTS

I am heartily thankful to my advisor Mark van Hoeij, whose encouragement, guidance and support enabled me to research on the subject. He has been a great friend and a mentor beyond the supervisor of the dissertation. Moreover, I owe my deepest gratitude to his wife Susan, who has been dearest friend of our family.

It is a pleasure to thank all committee members for their time and effort. Lastly, I offer my regards and blessings to the staff members of the Department of Mathematics at Florida State University. They have supported me in any respect during the completion of the thesis.

TABLE OF CONTENTS

Abstract	vii
1 Introduction	1
2 Preliminaries	3
2.1 Sequences, Difference Equations	3
2.2 Difference Rings, Ring of Difference operators, Difference Modules	4
2.3 Equivalence Relations	7
2.3.1 Gauge Equivalence	7
2.3.2 Term Transformation	7
3 Local Data	10
3.1 Valuation Growth	10
3.1.1 Definitions	11
3.1.2 Main Theorem	13
3.2 Generalized Exponents	13
3.2.1 Indicial Equation	14
3.2.2 Integer roots of the indicial equation	15
3.2.3 Roots of the indicial equation	17
3.2.4 Generalized Exponents	18
3.2.5 Generalized exponents, Unramified Case	19
3.2.6 Generalized exponents, Ramified Case	20
3.2.7 Computing Generalized Exponents	20
4 Liouvillian Solutions	24
4.1 Definitions	24
4.2 Main Theorem	25
4.2.1 Solutions of $\tau^n + c\phi$	28
4.3 Algorithms for order 2 and 3	28
4.4 Example	30
5 Special functions	32
5.1 Examples	32
5.2 The Table	34
5.3 Algorithm	35
5.4 Effectiveness	36

Bibliography	38
Biographical Sketch	40

ABSTRACT

In this thesis we present an algorithm that finds closed form solutions for homogeneous linear recurrence equations. The key idea is transforming an input operator L_{inp} to an operator L_g with known solutions. The main problem of this idea is how to find a solved equation L_g to which L_{inp} can be reduced. To solve this problem, we use local data of a difference operator, that is invariant under the transformation.

CHAPTER 1

INTRODUCTION

The object of this thesis is to present an algorithm ‘*solver*’ that gives closed form solutions of linear difference equations. Let τ be a shift operator that acts on $\mathbb{C}(x)$ by $\tau(x) = x + 1$. Let $V(L)$ denote the solution space of a difference operator $L \in \mathbb{C}(x)[\tau]$ and suppose $L_1, L_2 \in \mathbb{C}(x)[\tau]$ are of same order, then we define the following transformations.

- (a) $G \in \mathbb{C}(x)[\tau]$ is called a gauge transformation if $G(V(L_1)) = V(L_2)$.
- (b) $T(t) = rt$ is called a term transformation if $T(V(L_1)) = V(L_2)$ for some hypergeometric term r .

We say L_1 and L_2 are GT-equivalent if there is a combination of (a) and (b) that maps $V(L_1)$ to $V(L_2)$. Suppose we know the solutions of an operator L_1 , and suppose that L_1 is GT-equivalent to L_2 , then by applying the GT-transformation, we obtain solutions of L_2 .

The main idea of *solver* is to reduce an input operator L_{inp} to an operator L_g with known solutions which is GT-equivalent to L_{inp} . The basic strategy is:

1. Construct a database of operators with known solutions (can be parameterized family of operators).
2. Compute data which are invariant under GT-transformations.
3. Compare the data, select L_g from the database, and reconstruct the parameters in L_g to obtain one member of the parameterized family.
4. Find GT-transformation, if it exists, and apply it to the known solution of L_g .

We can collect base equations by looking up previous work, [5] and [18]. In [1], recurrence relations and closed form solutions are given for many sequences. We can build an operator (with parameters) with those relations. Also, there are algorithms that returns recurrence relations for an input function, called Gosper and Zeilberg algorithm.

The GT-transformation (if it exists) can be computed by the algorithm in [7] or [17, Chapter 3] but only after substituting the correct parameter values into the right base equation. Thus, the main problem of the process will be computing data that is invariant under GT-transformations and using it to reconstruct the parameter values. To compute GT-invariant data we compute local data. Local data of a difference operator are valuation growths at finite singularities in \mathbb{C}/\mathbb{Z} and generalized exponents at the point at infinity.

In Chapter 3, we will discuss about local data. Valuation growth is an idea introduced in [25], and the mathematics of generalized exponents has been treated in [21]. New in this thesis is not the concept of local data rather how it is been used. We will prove that these data are invariant under GT-transformation, Theorem 3.1.9 and Theorem 3.2.14. Also, we will introduce shorter proof of known facts, Lemma 3.2.8 and Lemma 3.2.9. In Section 3.2.7, we give an algorithm that computes generalized exponents. Computing in a simpler case, the unramified case, was explained in [15]. We will present an algorithm in more general case.

The first base equation we had in *solver* was equations of the form $\tau^n - \phi$, such equations are said to have Liouvillian solutions. This base equation is parameterized by rational function $\phi \in \mathbb{C}(x)$. So, to solve difference operators those are GT-equivalent to this base equation, we need to find $\phi \in \mathbb{C}(x)$. In our paper [13], we have solved this problem with valuation growths. There were algorithms solving these kind of base equation but we have reduced the combinatorial problem. Chapter 4 is Section 3, 4 and 5 of [13].

Chapter 5 is extension of our publication [14]. In [14] we have used generalized exponent, in addition to valuation growths, to add more base equations to *solver*. There were only 5 base equations in [14], but in Chapter 5 there are more base equations. Also, we give table of base equations and their local data and, show effectiveness of *solver*.

An advantage of *solver* is that whenever we find a new base equation, we can easily add that equation to *solver*. We simply compute the local data and add those in to the table. In this thesis most of base equations in the table are of order 2, only one order of 3, $\tau^3 - \phi$. However, the same method can be extended to order > 3 .

CHAPTER 2

PRELIMINARIES

This chapter will introduce well-known definitions and basic knowledge about the subject from [11], [12], [13], [14], [15], [17], [25], [19], [22], [23] and [24]. Also, we will fix the notations that is needed for later chapters. Readers familiar with the subject can skip to Chapter 3.

2.1 Sequences, Difference Equations

A sequence in \mathbb{C} is a function $f : \mathbb{N} \rightarrow \mathbb{C}$ and we will denote it as $(f(1), f(2), \dots)$. Let $\mathbb{C}^{\mathbb{N}}$ be the set of all sequence in \mathbb{C} . Then $\mathbb{C}^{\mathbb{N}}$ is a commutative ring by defining addition and multiplication termwise and it is also a \mathbb{C} -algebra by defining $c \cdot f(x)$ termwise for $c \in \mathbb{C}$ and $f \in \mathbb{C}^{\mathbb{N}}$.

Let $DE : \mathbb{C}^{n+2} \rightarrow \mathbb{C}$. Then a difference equation is an equation of the form

$$DE(f(x), f(x+1), \dots, f(x+n), x) = 0 \quad (n \geq 1)$$

and constant n is said to be the order of the difference equation. We say a sequence $u \in \mathbb{C}^{\mathbb{N}}$ is a solution of a difference equation if it makes the equation valid.

A recurrence relation is a special case of difference equation. Let $R : \mathbb{C}^{n+1} \rightarrow \mathbb{C}$ then a recurrence relation is an equation of the form

$$f(x+n) = R(f(x), f(x+1), \dots, f(x+n-1), x) \quad (n \geq 1)$$

If difference equation is linear that is, if it is in the form of

$$a_n(x)f(x+n) + a_{n-1}(x)f(x+n-1) + \dots + a_0(x)f(x) + a(x) = 0$$

where $a, a_i : \mathbb{C} \rightarrow \mathbb{C}$ for $i = 0, \dots, n$ then it naturally defines a recurrence relation by

$$f(x+n) = -\frac{a_{n-1}(x)}{a_n(x)}f(x+n-1) - \dots - \frac{a_0(x)}{a_n(x)}f(x) - \frac{a(x)}{a_n(x)}. \quad (2.1)$$

A difference equation or recurrence relation is called homogeneous if $a(x) = 0$ and inhomogeneous otherwise. In this paper we will only consider homogeneous linear difference equations with coefficients in $\mathbb{C}(x)$.

Example 2.1.1. Many special functions satisfy differential equations, and difference equations w.r.t. their parameters.

1. Let $P_x(z)$ be the Legendre function. Then it is defined recursively by

$$P_{x+2}(z) = \frac{(2x+3)(x+1)}{x+2}P_{x+1}(z) - \frac{x+1}{x+2}P_x(z)$$

and the corresponding difference equation is

$$(x+2)f(x+2) - (2x+3)(x+1)f(x+1) + (x+2)f(x) = 0. \quad (2.2)$$

2. All types of Bessel functions satisfies the recurrence relation

$$B_{x+2}(z) = \frac{2x+2}{z}B_{x+1}(z) - B_x(z)$$

where $B_x(z)$ denotes a Bessel function and the corresponding difference equation is

$$zf(x+2) - (2x+2)f(x+1) + (2x+2)f(x) = 0.$$

A function is said to be in a closed form if it is linear combination of elementary functions. In this paper we will extend the definition of closed form to be a function that is linear combination of elementary functions and special functions over $\mathbb{C}(x)$. Thus, Legendre function is a closed form solution of difference equation (2.2) in the above example.

2.2 Difference Rings, Ring of Difference operators, Difference Modules

Definition 2.2.1. A difference ring is a commutative ring R , with 1, together with an automorphism $\tau : R \rightarrow R$. If R is a field then we say R is a difference field. The constants of a difference ring R are the elements $c \in R$ satisfying $\tau(c) = c$. A difference ideal of a difference ring is an ideal I such that $\tau(a) \in I$ for all $a \in I$. A simple difference ring is a difference ring R whose only difference ideal are (0) and R .

Example 2.2.2. The following are a difference ring and difference fields with corresponding automorphism.

- We say two sequences $u, v \in \mathbb{C}^{\mathbb{N}}$ are equivalent, $u \sim v$, if there exist $N \in \mathbb{N}$ such that

$$u(x) = v(x) \quad \text{for all } x \geq N.$$

Let $\mathcal{S} := \mathbb{C}^{\mathbb{N}} / \sim$ then \mathcal{S} is a difference ring with automorphism $\tau((f(1), f(2), f(3), \dots)) = (f(2), f(3), \dots)$. Then τ is injective and \mathcal{S} is not simple.

- $k := \mathbb{C}(x)$ with automorphism $\tau(x) = x + 1$.
- $K := \mathbb{C}((t))$, the formal power series in $t = x^{-1}$ with automorphism $\tau(t) = \frac{t}{1+t}$.

- \overline{K} , the algebraic closure of $\mathbb{C}((t))$ with automorphism $\tau(t^{\frac{1}{m}}) = t^{\frac{1}{m}}(1+t)^{-\frac{1}{m}}$, $m \in \mathbb{Z}$.

The ring of linear difference operator $D = \mathbb{C}(x)[\tau]$ forms a noncommutative ring with $\tau \cdot a = \tau(a)\tau$ for $a \in \mathbb{C}(x)$ and it is both left and right Euclidean. We say $L = a_n\tau^n + \dots + a_0\tau^0$ is called normal if $a_0 \neq 0$. With extended right Euclidean algorithm we can compute gcd and lcm for L_1 and $L_2 \in D$.

Extended right Euclidean algorithm on D (see [11, 12, 23] for more detail)

Let L_1 and $L_2 \in D$ and suppose $\text{ord}(L_1) \geq \text{ord}(L_2)$ then there is Q and $R \in D$ where $\text{ord}(R) < \text{ord}(L_2)$ that satisfies

$$L_1 = QL_2 + R.$$

Then R is called the right remainder of L_1 by L_2 and denoted by $\text{rrem}(L_1, L_2)$, and Q is called the right quotient of L_1 by L_2 and denoted by $\text{rquo}(L_1, L_2)$.

Algorithm 2.2.3. GCRD

Input: L_1 and $L_2 \in D$

Output: The great common right divisor of L_1 and L_2

1. $R_0 := L_1, R_1 := L_2$.
2. For $i \geq 2$ do $R_i := \text{rrem}(R_{i-2}, R_{i-1})$.
3. Stop if $R_i = 0$.
4. Return R_{i-1} .

Algorithm 2.2.4. LCLM

Input: L_1 and $L_2 \in D$

Output: The least common left multiple of L_1 and L_2

1. $R_0 := L_1, R_1 := L_2,$
 $A_0 := 1, A_1 := 0,$
 $B_0 := 0, B_1 := 1.$
2. For $i \geq 2$ do
 $Q_{i-1} := \text{rquo}(R_{i-2}, R_{i-1}),$
 $R_i := \text{rrem}(R_{i-2}, R_{i-1}),$
 $A_i := A_{i-2} - Q_{i-1}A_{i-1},$
 $B_i := B_{i-2} - Q_{i-1}B_{i-1}.$
3. Stop if $R_i = 0$
4. Return $A_n A$

Definition 2.2.5. Let $k = \mathbb{C}(x)$. A k -algebra V is called a universal extension of k if the following three conditions hold

- $\tau : V \rightarrow V$ is an automorphism that extends $\tau : k \rightarrow k$.

- For every $L \in D$ the kernel of $L : V \rightarrow V$ is an $\text{ord}(L)$ -dimensional \mathbb{C} -vector space.
- For every $u \in V$ there exists a non-zero $L \in D$ such that $L(u) = 0$.

Existence and uniqueness of such extension are proved in [24, Chapter 1]. There are several ways to construct such V . One way is $\{u \in \mathcal{S} \mid \exists L \in D, L \neq 0, L(u) = 0\}$. LCLM resp., symmetric product (see Definition 2.3.4) show that this set is closed under $+$ resp., \cdot . We denote $V(L)$ as solution space of $L \in D$ in V , where V is a universal extension.

Lemma 2.2.6. [25, Lemma1] For every non-zero $M \in D$ there exists a unique monic normal operator $L \in D$ for which $V(L) = V(M)$. If $L_1, L_2 \in D$ are normal operators then

- L_1 is a right-hand factor of L_2 if and only if $V(L_1) \subseteq V(L_2)$.
- $L_3 = \text{GCRD}(L_1, L_2)$ is normal and $V(L_3) = V(L_1) \cap V(L_2)$.
- $L_4 = \text{LCLM}(L_1, L_2)$ is normal and $V(L_4) = V(L_1) + V(L_2)$.

In this paper, we will always assume operators to be normal, or normal and monic by the above Lemma.

Definition 2.2.7. Let $L = \sum_{i=0}^n a_i \tau^i$, $a_i \in \mathbb{C}(x)$ and $a_n = 1$ be linear difference operator. Then we can form the system $\tau(Y) = A_L Y$ where

$$A_L = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}.$$

We call the matrix A_L the companion matrix of the equation $L(y) = 0$.

So, y is solution of $L(y) = 0$ if and only if

$$Y = (y, \tau(y), \dots, \tau^{n-1}(y))^T$$

is solution of $\tau(Y) = A_L Y$.

Definition 2.2.8. A D -module is a k -vector space M on which action of τ has been defined, an action that is compatible with the ring structure of D . That means: a k -linear map $\Phi : M \rightarrow M$ satisfying $\Phi(am) = \tau(a)\Phi(m)$ for all $a \in k, m \in M$. We will only consider left modules M over D that are finite dimensional as a k -vector space. Such a module is called a difference module over k or a D -module.

A D -module M can be turned into a system A by choosing a basis.

Example 2.2.9. Let $L = \tau^n + a_{n-1}\tau^{n-1} + \dots + a_0 \in D$, then we denote $M_L = D/DL$ as the D -module given by L . A basis of M_L is $\{1, \tau, \tau^2, \dots, \tau^{n-1}\}$ and the action of τ on M_L is given by $\tau \cdot \tau^i = \tau^{i+1}$ for $i < n-1$ and $\tau \cdot \tau^{n-1} = -a_{n-1}\tau^{n-1} - \dots - a_0$.

2.3 Equivalence Relations

2.3.1 Gauge Equivalence

Definition 2.3.1. Two operators L_1 and L_2 in $\mathbb{C}(x)[\tau]$ are called gauge equivalent if the D -modules given by L_1 and L_2 are isomorphic. In this case we write $L_1 \sim_g L_2$.

Lemma 2.3.2. Suppose $M_{L_1} \cong M_{L_2}$. Let $G_1 \in M_{L_1}$ be the image of $1 \in M_{L_2}$ in M_{L_1} . Let $G_2 \in M_{L_2}$ be the image of $1 \in M_{L_1}$ in M_{L_2} . Then G_1 and G_2 define bijective \mathbb{C} -linear maps $G_1 : V(L_1) \rightarrow V(L_2)$ and $G_2 : V(L_2) \rightarrow V(L_1)$, respectively, that are inverse to each other. The linear map $G_1 : V(L_1) \rightarrow V(L_2)$ is called a gauge transformation from L_1 to L_2 .

Definition 2.3.3. Let V be a universal extension of k . A non-zero element $u(x) \in V$ is called hypergeometric term if it is a solution for some $\tau - r(x) \in D$. Then $r(x) = u(x+1)/u(x)$ and it is called the certificate of $u(x)$. Let $u_1(x), u_2(x)$ be hypergeometric terms, and $r_1(x) = u_1(x+1)/u_1(x)$, $r_2(x) = u_2(x+1)/u_2(x)$ be their certificates. Then $\tau - u_1(x)$ and $\tau - u_2(x)$ are gauge equivalent if $u_1(x)/u_2(x)$ is a rational function, or equivalently, if and only if $r_1(x)/r_2(x)$ is the certificate of a rational function.

Suppose two linear difference operators L_1 and L_2 are gauge equivalent. Then there is a gauge transformation G from the solutions of L_1 to the solutions of L_2 . Let A_{L_1}, A_{L_2} be companion matrices for L_1, L_2 and A_G be corresponding matrix for G such that multiplying by A_G is a bijection from the solution space of $\tau(Y) = A_{L_1}Y$ to the solution space of $\tau(Z) = A_{L_2}Z$. Then

$$\begin{aligned}\tau(Z) &= A_{L_2}Z \\ \tau(A_G Y) &= A_{L_2}A_G Y \\ \tau(A_G)\tau(Y) &= A_{L_2}A_G Y \\ \tau(Y) &= \tau(A_G)^{-1}A_{L_2}A_G Y\end{aligned}$$

Thus

$$A_{L_1} = \tau(A_G)^{-1}A_{L_2}A_G. \tag{2.3}$$

Equation 2.3 can be another definition of gauge equivalence relation, that is, we can define L_1 and L_2 are gauge equivalent if there is a matrix $A_G \in Gl_n(k)$ that satisfies equation 2.3. If such matrix exist then the D -modules given by L_1 and L_2 are isomorphic.

2.3.2 Term Transformation

Definition 2.3.4. Let $L_1, L_2 \in \mathbb{C}(x)[\tau]$. The symmetric product of L_1 and L_2 written $L_1 \otimes L_2$ is defined as the monic operator $L \in \mathbb{C}(x)[\tau]$ of minimal order such that $L(u_1 u_2) = 0$ for all u_1, u_2 with $L_1(u_1) = 0$ and $L_2(u_2) = 0$. For the case $L_2 = \tau - r$ with $r \in \mathbb{C}(x)$ we call $\otimes L_2$ a term transformation which is an automorphism of $\mathbb{C}(x)[\tau]$.

The formula for a term transformation is

$$L \otimes (\tau - r) = \sum_{i=0}^n b_i \tau^i \tag{2.4}$$

where $b_n = a_n$ and $b_i(x) = a_i(x) \prod_{j=i}^{n-1} \tau^j(r(x))$ for $i = 0, \dots, n-1$.

Given a series of gauge and term transformations from one operator to another, the following theorems reduce the problem of finding those transformations to that of finding exactly one gauge and one term transformation.

Theorem 2.3.5. [Theorem 3.3 [17]] Let s_1, \dots, s_m be some combination of gauge transformations and term transformations. A transformation $L_1 \xrightarrow{s_1 \circ \dots \circ s_m} L_2$ can be written $L_1 \xrightarrow{t_2 \circ t_1} L_2$ for some term transformation t_1 and some gauge transformation t_2 .

Definition 2.3.6. $L_1 \xrightarrow{t_2 \circ t_1} L_2$, for some term transformation t_1 and some gauge transformation t_2 , will be called a GT-transformation. We say L_1 and L_2 are GT-equivalent if there is a GT-transformation from L_1 to L_2 .

Definition 2.3.7. Let $\hat{r}(x) = cp_1(x)^{e_1} \dots p_j(x)^{e_j} \in C(x)$ with $C \subseteq \mathbb{C}$. Let the $e_i \in \mathbb{Z}$, let the $p_i(x)$ be irreducible in $C[x]$, and let $s_i \in C$ equal the sum of the roots of $p_i(x)$. $\hat{r}(x)$ is said to be in shift normal form if $-\deg(p_i(x)) < \operatorname{Re}(s_i) \leq 0$, for $i = 1, \dots, j$. We denote $\operatorname{SNF}(r(x))$ as the shift normalized form of $r(x)$ which is obtained by replacing each $p_i(x)$ by $p_i(x + k_i)$ for some $k_i \in \mathbb{Z}$ such that $p_i(x + k_i)$ is in shift normal form.

Let $L = a_n \tau^n + \dots + a_0$, $a_n \neq 0$. Then we denote $\det(L)$ by determinant of companion matrix of L which is $(-1)^n a_0/a_n$. Then the following lemma comes from equation 2.3.

Lemma 2.3.8. If $L_1 \sim_g L_2$ then $\operatorname{SNF}(\det(L_1)) = \operatorname{SNF}(\det(L_2))$.

In [17, Theorem 3.4], the following theorem is only proven for order 2. Here we give a proof for arbitrary order n .

Theorem 2.3.9. Let $L_1, L_2 \in D$ with same order n and the leading and trailing coefficient of L_1 be non-zero. If $L_1 \xrightarrow{G \circ T} L_2$ for some gauge transformation G and some term transformation T then there exists a gauge transformation $L_1 \otimes (\tau - r) \rightarrow L_2$ for some $r \in \mathbb{C}(x)$ where

$$r^n = \operatorname{SNF}(\det(L_2)/\det(L_1)).$$

Proof. Let $\otimes(\tau - \tilde{r})$ be the term transformation T and $r = \operatorname{SNF}(\tilde{r})$. Then there is a gauge transformation $G = q(x) \in \mathbb{C}(x)$ such that $G(V(\tau - \tilde{r})) = V(\tau - r)$. Thus, $L_1 \otimes (\tau - \tilde{r}) \sim_g$

$L_1 \otimes (\tau - r) \sim_g L_2$. By equation 2.4, $\det(L_1 \otimes (\tau - r)) = (-1)^n \frac{a_0(x)}{a_n(x)} \prod_{j=0}^{n-1} \tau^j(r)$. Hence, by

above lemma $\operatorname{SNF}(\det(L_1))r^n = \operatorname{SNF}(\det(L_2))$. \square

Definition 2.3.10. It is in the context of Theorem 2.3.9 that we say that L_2 can be reduced to L_1 .

We provide an algorithm from [17] that finds such a reduction if it exists.

Algorithm 2.3.11. Find GT-Transformation

Input: $L_1, L_2 \in \mathbb{C}[x][\tau]$ linear difference operators of order 2.

Output: Operator of the form $H(x)(c_1(x)\tau + c_0(x))$ mapping $V(L_1)$ to $V(L_2)$ if it exists, where $c_0, c_1 \in \mathbb{C}(x)$ and $H(x)$ is a hypergeometric term.

1. Calculate $\hat{r} = \text{SNF}(\det(L_2)/\det(L_1))$.
2. If \hat{r} is a square in $\mathbb{C}(x)$ then let $r = \sqrt{\hat{r}}$ else return 'FAIL' and stop.
3. Calculate $L_{neg} = L_1 \otimes (\tau - r)$ and $L_{pos} = L_1 \otimes (\tau + r)$.
4. Compute a gauge transformation, $c_1(x)\tau + c_0(x)$, between L_{neg} and L_2 (see [7] or [17]).
 - (a) If a gauge transformation exists then return $H(x) \cdot (c_1(x)\tau + c_0(x))$ and exit, where $H(x)$ is a solution of $(\tau - r)$.
 - (b) If no gauge transformation exists then go to Step 5.
5. Compute a gauge transformation, $c_1(x)\tau + c_0(x)$, between L_{pos} and L_2 .
 - (a) If a gauge transformation exists then return $H(x) \cdot (c_1(x)\tau + c_0(x))$ and exit, where $H(x)$ is a solution of $(\tau + r)$.
 - (b) If no gauge transformation exists then return 'FAIL.'

Finding a gauge transformation can be reduced to finding a rational solution of a system of recurrence equations, which can be done with [7] or [4].

Example 2.3.12. Here we will check the above algorithm with operators in Example 5.1.2. Let $L_1 = -\frac{1}{3}(2+x)\tau^2 + (2 + \frac{4}{3}x)\tau - 1 - x$ and $L_2 = (x+4)\tau^2 + (-20 - 8x)\tau + (12x + 12)$. Then $\hat{r} = 4$ and $r = 2$. By computing gauge transformation between $L_1 \otimes (\tau - 2)$ and L_2 we get $\frac{1}{x+2}(-2\tau + 3)$. Thus the algorithm returns $2^x(\frac{1}{x+2}(-2\tau + 3))$.

CHAPTER 3

LOCAL DATA

In this thesis I have developed an algorithm to solve a linear difference operator called *solver*. The idea to solve an input operator L_{inp} is as follows; find an operator L_g , from a table of base equations with known solutions, that is GT-equivalent to L_{inp} .

The main problem is: given L_{inp} , how can we find such an operator L_g ? The issue is that there are numerous base equations with known solutions that contain a number of parameters. The GT-transformation can only be computed (Algorithm 2.3.11) after substituting the correct parameter values into the right base equation. To compute these parameter values (and to select the right base equation from the table), we must compute GT-invariant data from L_{inp} . Only GT-invariant data computed from L_{inp} can help us to find the parameter values for which L_{inp} can be matched (up to GT-equivalence) to a base equation in the table.

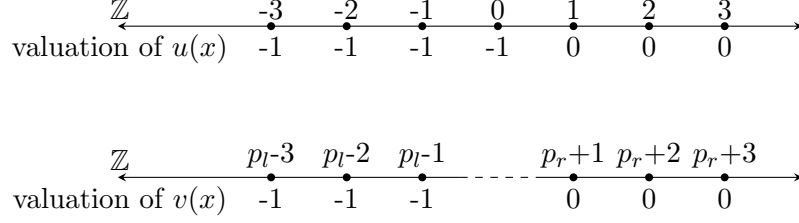
GT-invariant data that solves the main problem is local data. Local data of a difference operator are valuation growths at finite singularities in \mathbb{C}/\mathbb{Z} and generalized exponents at the point at infinity. In [13] and [14], we have shown these data are invariant under GT-transformations. An algorithm that computes valuation growths is already in MAPLE and I have implemented code that computes generalized exponents. The program *solver* is a novel application of generalized exponents.

3.1 Valuation Growth

Valuation growth was first introduced in [25] and an algorithm was given in the same paper. The algorithm computes the set $\{\overline{g}_p(L) \mid p \text{ is an essential singularity of } L \text{ in } \mathbb{C}/\mathbb{Z}\}$ where $\overline{g}_p(L)$ is the set of valuation growth of L at $p \in \mathbb{C}/\mathbb{Z}$. An analog of $\overline{g}_p(L)$ is: let $f(x) = c_n x^n + \dots + c_0 \in \mathbb{Z}[x]$ and suppose r is a rational root of $f(x)$. Then $r \in \{\frac{p}{q} \mid p|c_0, q|c_n\}$. Like in polynomial case, if $u(x)$ is a hypergeometric solution of L then valuation of $u(x)$ at p is in $\overline{g}_p(L)$.

The idea of valuation growth are as follows: first we find a data that is invariant under gauge transformation and then adjust it to make it invariant under term transformation. Suppose $L_1 \sim_g L_2$ and let $G = r_k(x)\tau^k + \dots + r_0(x), r_i(x) \in \mathbb{C}(x)$ be the gauge transformation from $V(L_1)$ to $V(L_2)$. Suppose for example that $u(x) = \Gamma(x) \in V(L_1)$ and $G(u(x)) = v(x)$ is a non-zero element in $V(L_2)$. Then $v(x) = G(u(x)) = r_k(x)u(x+k) + \dots + r_0(x)u(x)$. What will be common between $u(x)$ and $v(x)$? $u(x)$ has a pole order of 1 (valuation -

1) at non-positive integers and no root or pole (valuation 0) at positive integers. Since $r_0(x), \dots, r_k(x)$ have finitely many roots or poles in \mathbb{Z} , there exists integers p_l and p_r such that for all integers less than p_l , $v(x)$ has a pole of order 1 (i.e. valuation -1) and for all integers greater than p_r , valuation is 0 (i.e. no root or pole).



So evaluating the valuation from left to right in \mathbb{Z} , passing through the problem points (see Definition 3.1.1), the valuation of $u(x)$ and $v(x)$ increases by 1. This data is called the valuation growth at $p = \mathbb{Z} \in \mathbb{C}/\mathbb{Z}$ and it is invariant under gauge transformation.

There is an algorithm and implementation that, given $L = a_n(x)\tau^n + \dots + a_0(x)$, can compute the set of valuation growth of all meromorphic solutions (see [18] and [15]). From equation 2.1, to calculate $f(\alpha), \alpha \in \mathbb{C}$ from the left, which means initial values are $f(\alpha - 1), \dots, f(\alpha - n)$, then we need to divide by $a_n(x)$ and to calculate from the right, which means initial values are $f(\alpha + 1), \dots, f(\alpha + n)$, we divide by $a_0(x)$. If $a_n(x)a_0(x)$ has a root in $\alpha + \mathbb{Z}$, then it leads us to dividing by 0. So, the algorithm substitute $x = x + \varepsilon$ and measures the order of ε to compute valuation growth.

3.1.1 Definitions

Let $L = a_n\tau^n + \dots + a_0\tau^0 \in \mathbb{C}(x)[\tau]$. After multiplying L on the left by a suitable element of $\mathbb{C}(x)$, we may assume that the a_i are in $\mathbb{C}[x]$ and $\gcd(a_0, \dots, a_n) = 1$.

Definition 3.1.1. Let $L = a_n\tau^n + \dots + a_0\tau^0$ with $a_i \in \mathbb{C}[x]$. $q \in \mathbb{C}$ is called a problem point of L if q is a root of the polynomial $a_0(x)a_n(x - n)$. $p \in \mathbb{C}/\mathbb{Z}$ is called a finite singularity of L if L has a problem point in p (i.e. $p = q + \mathbb{Z}$ for some problem point q).

Definition 3.1.2. Let $p \in \mathbb{C}/\mathbb{Z}$. For $a, b \in p \subset \mathbb{C}$ we say $a > b$ iff $a - b$ is a positive integer.

Let $L = \sum_{i=0}^n a_i(x)\tau^i$, $a_i(x) \in \mathbb{C}(x)$ with $a_0 \neq 0, a_n \neq 0$ be a difference operator. We define $L_\varepsilon = \sum_{i=0}^n a_i(x + \varepsilon)\tau^i$ which is substituting x by $x + \varepsilon$ in L . The map $L \mapsto L_\varepsilon$ defines an embedding (as non-commutative rings) of $\mathbb{C}(x)[\tau]$ in $\mathbb{C}(x, \varepsilon)[\tau]$, so if $L = MN$ then $L_\varepsilon = M_\varepsilon N_\varepsilon$.

Definition 3.1.3. Let $a \in \mathbb{C}(\varepsilon)$. The ε -valuation $v_\varepsilon(a)$ of a at $\varepsilon = 0$ is the element of $\mathbb{Z} \cup \infty$ defined as follows: if $a \neq 0$ then $v_\varepsilon(a)$ is the largest integer $m \in \mathbb{Z}$ such that $a/\varepsilon^m \in \mathbb{C}[[\varepsilon]]$, and $v_\varepsilon(0) = \infty$.

Let $p \in \mathbb{C}/\mathbb{Z}$. We denote

$$V_p(L_\varepsilon) = \{\tilde{u} : p \rightarrow \mathbb{C}(\varepsilon) \mid L_\varepsilon(\tilde{u}) = 0\}.$$

Choosing q_l, q_r in p . Let q_l be the smallest root (by the ordering from Definition 3.1.2) of $a_0(x)a_n(x-n)$ in p , so q_l is the smallest problem point in p . Likewise we define q_r to be the largest root of $a_0(x)a_n(x-n)$ in p . If p is not singularity, that is, if a_0 and a_n have no roots in p , then choose two arbitrary elements in p and define q_l, q_r to be those two elements.

Definition 3.1.4. For non-zero $\tilde{u} \in V_p(L_\varepsilon)$ and for $a, b \in \mathbb{C}$ if $b = a + n - 1$, we define the box-valuation

$$v_b^a(\tilde{u}) = \min\{v_\varepsilon(\tilde{u}(m)) \mid m = a, a + 1, \dots, b\}.$$

Lemma 3.1.5. With q_l, q_r chosen as above, we have

$$v_{q-1}^{q-n}(\tilde{u}) = v_{q_l-1}^{q_l-n}(\tilde{u}) \quad \text{for all } q \in \{q_l - 1, q_l - 2, q_l - 3, \dots\},$$

$$v_{q+n}^{q+1}(\tilde{u}) = v_{q_r+n}^{q_r+1}(\tilde{u}) \quad \text{for all } q \in \{q_r + 1, q_r + 2, q_r + 3, \dots\}.$$

Proof. We will only prove the first equation, the second equation can be proved likewise. Given two consecutive boxes $[q - n, \dots, q - 1]$ and $[q - (n - 1), \dots, q]$ the values of \tilde{u} at one box can be computed from the values of \tilde{u} at the other box using the relation $a_n(x + \varepsilon)\tilde{u}(x + n) + \dots + a_0(x + \varepsilon)\tilde{u}(x) = 0$ for $x = q - n$. This computation involves a division either by $a_n(q - n + \varepsilon)$ or by $a_0(q - n + \varepsilon)$. If $q \in \{q_l - 1, q_l - 2, q_l - 3, \dots\}$ then $a_n(q - n + \varepsilon)$ and $a_0(q - n + \varepsilon)$ have ε -valuation 0, and hence this division does not decrease the box valuation. So the valuation of each box can not be lower than the valuation of the other box, hence the boxes $[q - n, \dots, q - 1]$ and $[q - (n - 1), \dots, q]$ have the same box valuation. By repeating this one can check that the box valuation $v_{q-1}^{q-n}(\tilde{u})$ and $v_{q_l-1}^{q_l-n}(\tilde{u})$ must be equal for all $q \in \{q_l - 1, q_l - 2, q_l - 3, \dots\}$. \square

We define $v_{\varepsilon,l}(\tilde{u})$ as $v_{q_l-1}^{q_l-n}(\tilde{u})$ which, by Lemma 3.1.5, equals the box valuation of any box on the left of q_l . Likewise we define $v_{\varepsilon,r}(\tilde{u})$ as $v_{q_r+n}^{q_r+1}(\tilde{u})$.

Definition 3.1.6. Define the valuation growth of non-zero $\tilde{u} \in V_p(L_\varepsilon)$ as

$$g_{p,\varepsilon}(\tilde{u}) = v_{\varepsilon,r}(\tilde{u}) - v_{\varepsilon,l}(\tilde{u}) \in \mathbb{Z}.$$

Define the set of valuation growths of L at p as

$$\bar{g}_p(L) = \{g_{p,\varepsilon}(\tilde{u}) \mid \tilde{u} \in V_p(L_\varepsilon), \tilde{u} \neq 0\} \subset \mathbb{Z}.$$

Definition 3.1.7. Let L be a difference operator and $p \in \mathbb{C}/\mathbb{Z}$ be a finite singularity of L . If $\bar{g}_p(L) = \{0\}$ then p is called apparent singularity. If $\bar{g}_p(L)$ has more than one element then p is called essential singularity.

Note: this definition of apparent singularity is related, but not quite equivalent, to the definition in [3].

Lemma 3.1.8. If $p \in \mathbb{C}/\mathbb{Z}$ is not a finite singularity of L (i.e. if a_0 and a_n have no roots in p) then $\bar{g}_p(L) = \{0\}$.

Proof. Let $\tilde{u} \in V_p(L_\varepsilon)$, $\tilde{u} \neq 0$. Following the proof of Lemma 3.1.5 one can see that $v_{q-1}^{q-n}(\tilde{u})$ is the same for every $q \in p$. Hence the valuation growth of \tilde{u} is 0. \square

3.1.2 Main Theorem

Theorem 3.1.9. *If L_1 and L_2 are gauge equivalent then $\bar{g}_p(L_1) = \bar{g}_p(L_2)$ for every $p \in \mathbb{C}/\mathbb{Z}$.*

Proof. Let $G = c_{n-1}\tau^{n-1} + \dots + c_0 \in C(x)[\tau]$ be a gauge transformation from L_1 to L_2 and let $p \in \mathbb{C}/\mathbb{Z}$. Choose any non-zero $\tilde{u} \in V_p(L_{1,\varepsilon})$, and let $\tilde{v} = G_\varepsilon(\tilde{u})$ be the corresponding solution in $V_p(L_{2,\varepsilon})$. Then

$$\tilde{v}(q) = c_0(q + \varepsilon)\tilde{u}(q) + \dots + c_{n-1}(q + \varepsilon)\tilde{u}(q + n - 1) \quad (3.1)$$

for all $q \in p$. We can take $q' \in p$ such that for all $q \in \{q' - 1, q' - 2, q' - 3, \dots\}$

- (i) $q + 2n - 2$ is smaller (Definition 3.1.2) than any problem point of L_1 and L_2
- (ii) $c_0(q + \varepsilon), \dots, c_{n-1}(q + \varepsilon)$ have ε -valuation ≥ 0 .

If $q < q'$ then, by Lemma 3.1.5, $\min\{v_\varepsilon(\tilde{u}(m)) \mid m = q, \dots, q + (n-1)\} = v_{q+(n-1)}^q(\tilde{u}) = v_{\varepsilon,l}(\tilde{u})$ and by (ii) and Equation (3.1) $v_\varepsilon(\tilde{v}(q)) \geq v_{q+(n-1)}^q(\tilde{u})$. Hence $v_\varepsilon(\tilde{v}(q)) \geq v_{\varepsilon,l}(\tilde{u})$. Repeating this for $q, q+1, \dots, q+(n-1)$ we get $v_{\varepsilon,l}(\tilde{v}) = \min\{v_\varepsilon(\tilde{v}(m)) \mid m = q, \dots, q+(n-1)\} \geq v_{\varepsilon,l}(\tilde{u})$. Since gauge equivalence is an equivalence relation there is a gauge transformation $G' \in C(x)[\tau]$ from L_2 to L_1 . Using this G' we can get the opposite inequality $v_{\varepsilon,l}(\tilde{v}) \leq v_{\varepsilon,l}(\tilde{u})$. In all, $v_{\varepsilon,l}(\tilde{v}) = v_{\varepsilon,l}(\tilde{u})$. In the same way one can show $v_{\varepsilon,r}(\tilde{v}) = v_{\varepsilon,r}(\tilde{u})$. Thus, \tilde{v} and \tilde{u} have the same valuation growth, and hence $\bar{g}_p(L_1) = \bar{g}_p(L_2)$. \square

Lemma 3.1.10. *[25, Lemma 6] Let $\tilde{L} = L \otimes (\tau - a)$ for some $a \in \mathbb{C}(x)$ and let $\bar{g}_p(\tau - a) = \{d\}$ for some $d \in \mathbb{Z}$ and $p \in \mathbb{C}/\mathbb{Z}$. Then $\bar{g}_p(\tilde{L}) = \{n + d \mid n \in \bar{g}_p(L)\}$.*

Therefore term transformations do not preserve $\bar{g}_p(L)$ but they do preserve $d_p(L) = \max \bar{g}_p(L) - \min \bar{g}_p(L)$. We define the set

$$\text{Val}(L) = \{[p, d_p(L)] \mid p \in \mathbb{C}/\mathbb{Z} \text{ essential singularity of } L\}.$$

We compute $\text{Val}(L)$ (see our database in Section 5.2) because it is data that is invariant under GT -transformations.

3.2 Generalized Exponents

Let $K_r = \mathbb{C}((t^{1/r}))$ for some $r \in \mathbb{N}$ and let \bar{K} be the algebraic closure of K . Then

$$\bar{K} = \bigcup_{r \in \mathbb{N}} K_r.$$

A proof of this fact is given in [26, Theorem 3.1]. For the proof one needs to show that any $L \in K[y]$ of degree ≥ 1 has a linear factor in $K_r[y]$ for some r . The key to prove this is the Newton Polygon. A similar proof shows that any $L \in K[\tau]$ of order ≥ 1 has a first order right hand factor in $K_r[\tau]$. Write it as $\tau - a$ where $a = ct^v(1 + a_1t^{1/r} + a_2t^{2/r} + \dots)$, $c \in \mathbb{C}^*$, $v \in \frac{1}{r}\mathbb{Z}$. Let $S_a \in V(\tau - a) \subseteq V(L)$ with $S_a \neq 0$. The asymptotic behavior of S_a is determined by c, v, a_1, a_2, \dots . A gauge transformation can modify the asymptotic behavior by only a

limited amount (recall that a gauge transformation is composed of shifts, multiplying by rational functions and additions). Under a gauge transformation $c, v, a_1, \dots, a_{r-1}$ do not change at all and a_r changes only mod $\frac{1}{r}\mathbb{Z}$. A generalized exponent is defined to be the dominant part of a which is $ct^v(1 + a_1t^{1/r} + a_2t^{2/r} + \dots + a_r t^{r/r})$. Two generalized exponents are called equivalent if and only if all terms except possibly a_r coincide and the a_r 's differ only by a element of $\frac{1}{r}\mathbb{Z}$.

Claim If two difference operators are gauge equivalent then their generalized exponents match up to this equivalence.

This claim can be explained with formal solutions of a difference equation as in [8], [6], [9], [15], and [21]. Here we will prove the claim (Theorem 3.2.14) by associating generalized exponents with D -modules.

The main object of the section is to prove Theorem 3.2.14. In section 3.2.2 we treat the case when $r = 1, c = 1, v = 0, a_1 \in \mathbb{Z}$ and in section 3.2.3 the case when $r = 1, c = 1, v = 0, a_1 \in \mathbb{C}$. In section 3.2.5 we treat the case when $r = 1, c \in \mathbb{C}, v \in \mathbb{Z}, a_1 \in \mathbb{C}$ and in section 3.2.6 we handle the general case where $r \in \mathbb{N}, c \in \mathbb{C}, v \in \frac{1}{r}\mathbb{Z}, a_1 \in \mathbb{C}$

In our *solver*, given an input L_{inp} our aim is to construct an equation L_g for which L_{inp} is GT-equivalent to L_g , and, for which L_{inp} belongs to a parameterized class of equations with known solutions. By computing generalized exponents of L_{inp} , and adjusting them (see section 3.2.6) to make them invariant under term transformation, we obtain the data needed to find L_g .

3.2.1 Indicial Equation

The field $K = \mathbb{C}((t))$, $t = x^{-1}$ has a natural valuation $v : K \rightarrow \mathbb{Z} \cup \{\infty\}$ where $v(0) := \infty$ and

$$v(c_n t^n + c_{n+1} t^{n+1} + \dots) = n \text{ if } c_n \neq 0.$$

Let $\Delta := \tau - 1$, then $\mathbf{D} = K[\tau] = K[\Delta]$. Let $L \in \mathbf{D}$ and write $L = \sum_{i=0}^d a_i \Delta^i$. Now we extend the definition of v to \mathbf{D} as follows

$$v(L) := \min\{v(a_i) + i \mid i = 0, \dots, d\}.$$

Note that this $v : \mathbf{D} \rightarrow \mathbb{Z} \cup \{\infty\}$ still satisfies the properties of a valuation:

- (i) $v(L) = \infty \iff L = 0$,
- (ii) $v(L_1 + L_2) \geq \min\{v(L_1), v(L_2)\}$, (equality when $v(L_1) \neq v(L_2)$)
- (iii) $v(L_1 L_2) = v(L_1) + v(L_2)$ (follows from Corollary 3.2.3).

Lemma 3.2.1. *Let $L \in K[\tau]$. There exists a polynomial P such that for every $n \in \mathbb{Z}$ we have*

$$L(t^n) = P(n)t^{n+v(L)} + \dots \tag{3.2}$$

where the dots refer to terms of valuation $> n + v(L)$.

Proof. Let $\text{tc}(f)$ be the trailing coefficient of $f \in \mathbb{C}((t))$. $\Delta^i(t^n) = P_i(n)t^{n+i} + \dots$ where $P_i(n) = (-1)^i n(n+1) \dots (n+i-1)$ and $a_i \Delta^i(t^n) = P_i(n) \text{tc}(a_i) t^{n+i+v(a_i)} + \dots$. Let $M = \{i \in \mathbb{Z} \mid v(a_i) + i = v(L)\}$ then

$$L(t^n) = \sum_{i \in M} P_i(n) \text{tc}(a_i) t^{n+v(L)} + \dots$$

Then $P(n) = \sum_{i \in M} P_i(n) \text{tc}(a_i)$. □

Definition 3.2.2. Ind_L , the indicial equation of L , is the polynomial, $P(n)$, constructed in the proof of lemma 3.2.1.

3.2.2 Integer roots of the indicial equation

In this section we will consider various \mathbf{D} -modules. If M is such a module, and if $L \in \mathbf{D}$, then we get a \mathbb{C} -linear map

$$L : M \rightarrow M$$

We will denote the kernel of this map as

$$\text{Ker}(L, M).$$

We can interpret elements of $\text{Ker}(L, M)$ as solutions of L .

The first \mathbf{D} -module we shall consider is K . For this case we will prove lemma 3.2.4 below, but first some notation is needed.

Let $u \in K$, $u \neq 0$, and $v(u) = n$. Write

$$u = c_n t^n + c_{n+1} t^{n+1} + \dots$$

then it follows from equation (3.2) (use the fact that $L(a + b) = L(a) + L(b)$) that

$$L(u) = c_n P(n) t^{n+v(L)} + \dots$$

Corollary 3.2.3. Let $u \in K$, $u \neq 0$, then

$$v(L(u)) = v(u) + v(L) \iff v(u) \text{ is not a root of } \text{Ind}_L$$

Abbreviation: c.w.m. is an abbreviation for “counted with multiplicity”.

Lemma 3.2.4. Let $L \in K[\tau]$ and $L \neq 0$. Then

$$\dim(\text{Ker}(L, K)) > 0 \iff \text{mult}_{\mathbb{Z}}(\text{Ind}_L) > 0$$

where Ind_L denotes the indicial equation of L , and $\text{mult}_{\mathbb{Z}}(\text{Ind}_L)$ denotes the number (c.w.m.) of integer roots of Ind_L .

Proof. “ \implies ” if $u \in K$, $u \neq 0$, and $L(u) = 0$ then $v(u)$ must be a root of Ind_L by Corollary 3.2.3.

“ \impliedby ” Let n be the largest integer root of Ind_L , so

$$\text{Ind}_L(n) = 0, \text{Ind}_L(n+1) \neq 0, \text{Ind}_L(n+2) \neq 0, \dots \tag{3.3}$$

Since $\text{Ind}_L(n) = 0$ it follows from equation (3.2) that

$$L(t^n) = t^{n+v(L)} \cdot (0t^0 + a_1 t^1 + a_2 t^2 + \dots)$$

Write

$$u = t^n + c_{n+1}t^{n+1} + c_{n+2}t^{n+2} + \dots$$

Then write

$$L(u) = t^{n+v(L)} \cdot (0t^0 + A_1t^1 + A_2t^2 + \dots)$$

Now $A_1 = a_1 + c_{n+1}\text{Ind}_L(n+1)$ and since $\text{Ind}_L(n+1) \neq 0$ there is a unique $c_{n+1} \in \mathbb{C}$ for which A_1 vanishes, namely $c_{n+1} := -a_1/\text{Ind}_L(n+1)$. Then A_2 equals some constant plus $c_{n+2}\text{Ind}_L(n+2)$, and again $\text{Ind}_L(n+2) \neq 0$ so there is a unique c_{n+2} for which A_2 vanishes. Continuing this way leads to $L(u) = 0$. \square

Lemma 3.2.5. *Let $L = L_1L_2$. Then*

$$\text{Ind}_L(n) = \text{Ind}_{L_1}(n + v(L_2)) \cdot \text{Ind}_{L_2}(n)$$

Proof.

$$\begin{aligned} L(t^n) &= L_1(L_2(t^n)) = L_1(\text{Ind}_{L_2}(n)t^{n+v(L_2)} + \dots) \\ &= \text{Ind}_{L_2}(n)L_1(t^{v(L_2)+n}) + \dots \\ &= \text{Ind}_{L_2}(n)\text{Ind}_{L_1}(n + v(L_2))t^{v(L_1)+v(L_2)+n} + \dots \end{aligned}$$

\square

The following example shows that $\dim(\text{Ker}(L, K))$ does not need to be equal to $\text{mult}_{\mathbb{Z}}(\text{Ind}_L)$, and for this reason we will introduce a new module in definition 3.2.7 below.

Example 3.2.6. *Let $\Delta = \tau - 1$, let $L_1 = \Delta \cdot \frac{1}{t}$, and let $L = L_1 \cdot \Delta$. Then $\text{Ker}(L, K) = \mathbb{C}$ and $\text{Ind}_L(\lambda) = \lambda^2$. So $\dim(\text{Ker}(L, K)) = 1$ and $\text{mult}_{\mathbb{Z}}(\text{Ind}_L) = 2$.*

The above example was constructed by exploiting the fact that the map

$$\Delta : K \rightarrow K$$

is not onto, specifically, $t \notin \Delta(K)$. Thus, applying the right-factor Δ of $L = L_1\Delta$ to elements of $\text{Ker}(L, K)$ does not produce the solution t of the left-factor L_1 . This caused $\dim(\text{Ker}(L, K))$ to be lower than $\text{mult}_{\mathbb{Z}}(\text{Ind}_L)$.

Definition 3.2.7. *We turn the polynomial ring $K[l]$ into a \mathbf{D} -module by defining*

$$\tau(l) := l + t.$$

In other words $\Delta(l) := t$. Taking L as in Example 3.2.6, we have $\text{Ker}(L, K[l]) = \mathbb{C} + \mathbb{C}l$.

Note: Our l is defined in terms of modules, and not in terms of functions or series. If a series $u(1), u(2), \dots$ satisfies the equation that defined l , so if $\Delta(u) = t$, then $u(n+1) = u(n) + 1/n$ and so $u(n)$ can be written as $\ln(n) + b(n)$ where $b(n)$ is bounded by a constant ($b(n)$ converges to $u(1) + \gamma$, where γ is Euler's constant).

Lemma 3.2.8. *The map $\Delta : K[l] \rightarrow K[l]$ is onto.*

Proof. It is left to the reader to verify that $\Delta(K + \mathbb{C}l) = K$. Next, if $u \in K[l]$ has degree d as a polynomial in l , then write $u = u_d l^d + u_{d-1} l^{d-1} + \dots + u_0$. Let $v_d \in K + \mathbb{C}l$ be a pre-image of u_d under Δ . Then $\Delta(v_d l^d) = \tau(v_d)(l+t)^d - v_d l^d = \tau(v_d)(l^d + \dots) - v_d l^d = \Delta(v_d)l^d + \dots = u_d l^d + \dots$, where the dots refer to lower degree terms. Hence $u - \Delta(v_d l^d)$ has degree $< d$. By induction on this degree, there exists $v \in K[l]$ with $\Delta(v) = u - \Delta(v_d l^d)$. Then $\Delta(v + v_d l^d) = u$. \square

Note: Let $U = \mathbb{C}[x, x^{-1}, \log(x)]$. The differentiation map $d/dx : U \rightarrow U$ is onto (this follows easily from techniques of the transcendental Risch algorithm). The same techniques were used to prove this lemma.

Let $\Psi : K[l] \rightarrow K[l]$ be map given by $\sum a_i l^i \rightarrow \sum a_i (l+1)^i$. The \mathbf{D} -module $K[l]$ was defined by $\Delta(l) = t$. Note, however, that $\Delta(l+1)$ is also t , and from this it follows that Ψ is an automorphism (as \mathbf{D} -module) of $K[l]$.

Lemma 3.2.9. *If $\text{Ker}(L, K[l]) \neq \{0\}$ then $\text{Ker}(L, K) \neq \{0\}$.*

Proof. Let $u \in \text{Ker}(L, K[l])$ and $u \neq 0$. Let d be the degree of u as a polynomial in l . If $d = 0$ then there is nothing to prove. Suppose that $d > 0$. We have $\Psi(u) \in \text{Ker}(L, K[l])$ because $L \in \mathbf{D}$ and Ψ is an automorphism of $K[l]$ as \mathbf{D} -module. Hence $\Psi(u) - u \in \text{Ker}(L, K[l])$. Note, however, that $\Psi(u) - u$ has degree $d-1$ as a polynomial in l . Repeating this, we find that there exist non-zero elements of $\text{Ker}(L, K[l])$ of degree $d, d-1, d-2, \dots, 0$. \square

Theorem 3.2.10. *Let $L \in \mathbf{D}$, $L \neq 0$. Then*

$$\dim(\text{Ker}(L, K[l]) = \text{mult}_{\mathbb{Z}}(\text{Ind}_L).$$

Proof. Let $d = \dim(\text{Ker}(L, K[l])$ and $m = \text{mult}_{\mathbb{Z}}(\text{Ind}_L)$. We will prove the theorem with induction on m .

If $m = 0$ then $\text{Ker}(L, K) = \{0\}$ by lemma 3.2.4 and then $d = 0$ by lemma 3.2.9.

If $m > 0$ then L has a non-zero solution $u \in K$ by lemma 3.2.4. Write $R = \Delta \cdot \frac{1}{u}$, which has solution space $\mathbb{C}u$. Now R must be a right-hand factor of L because $\mathbb{C}u$ is a subset of the solution space of L . Write $L = L_1 R$. Let $m_1 = \text{mult}_{\mathbb{Z}}(\text{Ind}_{L_1})$. Let $S_1 = \text{Ker}(L_1, K[l])$. Let $S = \{s \in K[l] \mid \Delta(s) \in S_1\}$ be the pre-image of S_1 under Δ . Now $\Delta : K[l] \rightarrow K[l]$ is an onto map between two \mathbb{C} -vectorspaces and has a kernel of dimension 1, and hence $\dim(S) = \dim(S_1) + 1$. Now $S = \text{Ker}(L_1 \cdot \Delta, K[l])$. Hence $uS = \text{Ker}(L, K[l])$.

$\text{Ind}(R)$ has one integer root, $v(u)$, and hence $m = m_1 + 1$ by lemma 3.2.5. So $m_1 < m$ and hence we may assume $\dim(S_1) = m_1$ by induction. Then $d = \dim(uS) = \dim(S) = \dim(S_1) + 1 = m_1 + 1 = m$. \square

3.2.3 Roots of the indicial equation

If $c \in \mathbb{C}$ then there is a natural way to turn the set

$$M^c := t^c K[l]$$

into a \mathbf{D} -module, as follows:

$$\begin{aligned} \tau(t^c) &= t^c (1+t)^{-c} \\ &= t^c \left(1 - \frac{1}{1!} ct + \frac{1}{2!} c(c+1)t^2 - \frac{1}{3!} c(c+1)(c+2)t^3 + \dots \right) \in t^c K. \end{aligned}$$

If $c - \tilde{c} \in \mathbb{Z}$ then there is an obvious isomorphism between M^c and $M^{\tilde{c}}$. We will then identify M^c with $M^{\tilde{c}}$. This way we can define M^c not only for $c \in \mathbb{C}$, but for $c \in \mathbb{C}/\mathbb{Z}$ as well.

Given $L \in K[\tau]$ it is easy to see (from the fact that $\tau(t^c) \in t^c K$) that the operator $L^c := \frac{1}{t^c} L t^c$ is again in $K[\tau]$. Furthermore, from the definition of the indicial equation it follows easily that the indicial equations of L and L^c differ only by a shift of c . So the roots of $\text{Ind}(L)$ in $c + \mathbb{Z}$ correspond to the roots of $\text{Ind}(L^c)$ in \mathbb{Z} , which in turn (see Theorem 3.2.10) correspond to solutions of L^c in $K[l]$. This way it follows that:

Theorem 3.2.11. *Let $L \in \mathbf{D}$, $L \neq 0$. Then*

$$\dim(\text{Ker}(L, t^c K[l])) = \text{mult}_{c+\mathbb{Z}}(\text{Ind}_L).$$

In other words, the dimension of the solutions in $t^c K[l]$ equals the number (c.w.m.) of roots of Ind_L in $c + \mathbb{Z}$.

We can now form the following \mathbf{D} -module

$$M^{\mathbb{C}} := \bigoplus_{c \in \mathbb{C}/\mathbb{Z}} M^c.$$

Since the total number (c.w.m.) of roots of Ind_L in \mathbb{C} is simply the degree, we get

Theorem 3.2.12. *Let $L \in \mathbf{D}$, $L \neq 0$. Then*

$$\dim(\text{Ker}(L, M^{\mathbb{C}})) = \text{degree}(\text{Ind}_L)$$

Finally, we mention that $M^{\mathbb{C}}$ has an obvious ring-structure, and that the set of constants (i.e. the elements u with $\tau(u) = u$) in $M^{\mathbb{C}}$ is just \mathbb{C} .

To find $\text{ord}(L)$ linearly independent solutions, the difference ring $M^{\mathbb{C}}$ suffices if and only if Ind_L has maximal possible degree (degree $\text{ord}(L)$). If the degree is less, then a larger module is needed.

3.2.4 Generalized Exponents

For each $r \in \mathbb{N}$ we denote $K_r = \mathbb{C}((t^{1/r}))$. The algebraic closure of K is $\bar{K} = \bigcup_{r \in \mathbb{N}} K_r$.

Define the action of τ on K_r :

$$\begin{aligned} \tau(t^{\frac{1}{r}}) &= t^{\frac{1}{r}}(1+t)^{-\frac{1}{r}} \\ &= t^{\frac{1}{r}} \left(1 - \frac{1}{1!} \frac{1}{r} t + \frac{1}{2!} \frac{1}{r} \left(\frac{1}{r} + 1 \right) t^2 \right. \\ &\quad \left. - \frac{1}{3!} \frac{1}{r} \left(\frac{1}{r} + 1 \right) \left(\frac{1}{r} + 2 \right) t^3 + \dots \right) \in K_r. \end{aligned} \tag{3.4}$$

Since we have defined the action of τ on K_r , we can now apply the formula for the term transformation in Equation (2.4) to $K_r[\tau]$. Let E_r and \tilde{G}_r be the following subset and subring, respectively, of K_r^* .

$$E_r = \left\{ a \in K_r^* \mid a = ct^v \left(1 + \sum_{i=1}^r a_i t^{i/r} \right), a_i \in \mathbb{C}, c \in \mathbb{C}^*, v \in \frac{1}{r} \mathbb{Z} \right\}$$

$$\tilde{G}_r = \left\{ a \in K_r^* \mid a = 1 + \sum_{i=r+1}^{\infty} a_i t^{i/r}, a_i \in \mathbb{C} \right\}$$

Now E_r is a set of representatives for K_r^*/\tilde{G}_r . The composition of the natural maps $K_r^* \rightarrow K_r^*/\tilde{G}_r \rightarrow E_r$ defines a natural map

$$\text{Trunc} : K_r^* \rightarrow E_r.$$

Let

$$G_r = \left\{ a \in K_r^* \mid a = 1 + \frac{m}{r}t + \sum_{i=r+1}^{\infty} a_i t^{i/r}, a_i \in \mathbb{C}, m \in \mathbb{Z} \right\} \supseteq \tilde{G}_r.$$

If $a, b \in E_r$ then we say $a \sim_r b$ when $a/b \in G_r$.

Note: $a \sim_r b$ if and only if $a_r \equiv b_r \pmod{\frac{1}{r}\mathbb{Z}}$ with a_r as in the definition of E_r and $a_i = b_i$ for $i < r$ and c, v match as well.

Definition 3.2.13. Let $a \in E_r$ for some $r \in \mathbb{N}$. We say that a is a generalized exponent of L with multiplicity $m \Leftrightarrow 0$ is a root of $\text{Ind}_{\tilde{L}}$ with multiplicity m where $\tilde{L} = L \otimes (\tau - \frac{1}{a})$. We denote $\text{genexp}(L)$ as the set of generalized exponents of L .

3.2.5 Generalized exponents, Unramified Case

For each $a \in K^*$ we introduce the symbol S_a . We turn the set $\{S_a \mid a \in K^*\}$ into a multiplicative group by defining $S_a S_b := S_{ab}$. Next, we turn the following set

$$S_a K[l]$$

into a \mathbf{D} -module by defining $\tau(S_a) := a S_a \in S_a K$.

Let G_1 be the following subgroup of K^*

$$G_1 = \left\{ a \in K^* \mid a = 1 + \sum_{i=1}^{\infty} a_i t^i, a_i \in \mathbb{C}, a_1 \in \mathbb{Z} \right\}.$$

In Lemma 3.2.4 we see that the equation $\tau(u) = au$ (note: this is the equation that defines the action of τ on S_a) has a solution $u \in K$ if and only if $a \in G_1$. In this case, it follows from Corollary 3.2.3 that $v(u) = -a_1$. The solution u becomes unique if we suppose that its $t^{v(u)}$ term has coefficient 1. This u provides a canonical isomorphism from $S_a K[l]$ to $K[l]$ as \mathbf{D} -modules, namely $S_a P \mapsto uP$. Likewise, if $a_1, a_2 \in K^*$, and if $a_1/a_2 \in G_1$, then there is a canonical isomorphism between the modules $S_{a_1} K[l]$ and $S_{a_2} K[l]$. Hence, we can define this module not only for $a \in K^*$ but for $a \in K^*/G_1$ as well. We denote this module as M_a .

Remark: If $c \in \mathbb{C}$ then the module M^c from section 3.2.3 is isomorphic to M_a if we take $a = 1 - ct$.

3.2.6 Generalized exponents, Ramified Case

The field \overline{K} also has natural valuation $v : \overline{K} \rightarrow \mathbb{Q} \cup \{\infty\}$ so that $v(0) = \infty, v(t^r) = r$ by extending valuation on K . Then this still satisfies the properties of a valuation. The action of τ on \overline{K} is defined

$$\tau(t^{1/r}) = t^{1/r}(1+t)^{-1/r}.$$

We say $a \in \overline{K}$ has ramification index $r \in \mathbb{N}$ if r is the smallest number that $a \in \mathbb{C}((t^{1/r}))$. Then for each $a \in \overline{k}^*$ with ramification index r , $K(a) \subset \mathbb{C}((t^{1/r}))$. Since $[K(a) : K] = r$, $K(a) = \mathbb{C}((t^{1/r}))$. Thus, $S_a K(a)[l]$ is also a \mathbf{D} -module. As in the unramified case, we can define $S_a K(a)[l]$ for $a \in K_r^*/G_r$. We denote this module as \overline{M}_a .

Theorem 3.2.14. *Suppose $L_1 \sim_g L_2$, then for each $a \in \text{genexp}(L_1)$ there is a $b \in \text{genexp}(L_2)$ such that $a \sim_r b$. (where r is minimal with $a \in E_r$)*

Proof. Let G be a gauge transformation from L_1 to L_2 and $a \in \text{genexp}(L_1)$. Then there is non-zero $u \in \overline{M}_a$ such that $u \in V(L_1)$. Since $G(V(L_1)) = V(L_2)$ and $G \in D$, $G(u) \in V(L_2)$ and $G(u) \in \overline{M}_a$. Thus, there is $b \in \text{genexp}(L_2)$ such that $a/b \in G_r$. \square

The above theorem says generalized exponents mod \sim_r are invariant under gauge transformations. Suppose $\text{ord}(L) = 2$ and let $\text{genexp}(L) = \{a_1, a_2\}$ and $\tilde{L} = L \otimes (\tau - \alpha)$ for some $\alpha \in K_r, r \in \mathbb{N}$. Then

$$\text{genexp}(\tilde{L}) = \{\text{Trunc}(\alpha a_1), \text{Trunc}(\alpha a_2)\}.$$

To obtain an expression that is invariant under the term transformations as well, we define the quotient of the generalized exponents.

Definition 3.2.15. *Suppose $\text{ord}(L) = 2$ and let $\text{genexp}(L) = \{a_1, a_2\}$ such that $v(a_1) \geq v(a_2)$. If $v(a_1) > v(a_2)$ then we define the set of quotient of the two generalized exponents as $\text{Gquo} = \{\text{Trunc}(a_1/a_2)\}$. If $v(a_1) = v(a_2)$ then we define*

$$\text{Gquo}(L) = \{\text{Trunc}(a_1/a_2), \text{Trunc}(a_2/a_1)\}.$$

3.2.7 Computing Generalized Exponents

An Example of computing a generalized exponent in the unramified case is explained in [15]. We will explain how to compute generalized exponent with the examples in general case. Two main tools to compute generalized exponents are the Newton τ -polygon and indicial equation.

Definition 3.2.16. *Let $L = \sum_{i=0}^n a_i \tau^i, a_i \in \mathbb{C}[t]$. Then the Newton τ -polygon of L is defined to be the lower convex hull of the set of points of $(i, v(a_i))$ and is denoted as $\text{Ngon}(L)$.*

If s is a slope of a side of $\text{Ngon}(L)$, let

$$M_s(L) = \{i \in \mathbb{Z} \mid (i, v(a_i)) \in \text{Ngon}(L) \text{ is on the side with slope } s\}.$$

Then we define

$$Np_s(L) = \sum_{i \in M_s(L)} t c(a_i) X^{i-m}$$

to be the corresponding Newton τ -polynomial, where $tc(f)$ is trailing coefficient of $f \in \mathbb{C}((t))$ and $m = \min M_s(L)$.

Note Newton Δ -polygon of L is Newton polygon of $L \in \mathbb{C}(x)[\Delta]$ where $\Delta = \tau - 1$. Rewrite $L = \sum_{i=0}^n a_i \tau^i$ as $\sum_{i=0}^n a_i \Delta^i$, the Δ -polygon is computed from the b_i in the same as the τ -polygon from the a_i .

The following algorithm is a modification of the algorithm given in [15, Section 5]

Algorithm 3.2.17. τ poly

Input: An operator $L = a_n \tau^n + \dots + a_0 \in \mathbb{C}[t][\tau]$, $a_0, a_n \neq 0$.

Output: The slopes of the Newton τ -polygon and corresponding Newton τ -polynomials.

1. Let $v_i :=$ valuation of a_i for $i = 0, \dots, n$.
2. $s := \max \left\{ \frac{v_n - v_i}{n - i} \mid i = 0, \dots, n - 1 \right\}$
3. $m := \min \left\{ i < n \mid \frac{v_n - v_i}{n - i} = s \right\}$.
4. $R := \emptyset$
5. $P := \sum_{i=m}^n tc(a_i) X^{i-m}$
6. $R := [s, P]$
7. Return $R \cup \tau$ poly($a_m \tau^m + \dots + a_0$) and stop.

The basic Strategy for computing generalized exponents is the following: Suppose $g := ct^v(1 + a_1 t^{1/r} + \dots + a_r t^{r/r})$ is a generalized exponent of a difference operator L , that is L has a right hand factor of $\tau - g$.

- (a) $-v$ is a slope of the sides of Newton τ -polygon of L and c a root of corresponding Newton polynomial,
- (b) a_1 is a root of Newton Δ -polynomial of $M_{1/r}$ of $L_{cv} := L \otimes (\tau - 1/(ct^v))$.
- (c) a_2 is a root of Newton Δ -polynomial of $M_{2/r}$ of $L \otimes (\tau - 1/ct^v(1 + a_1 t^{1/r}))$.
- (d) a_3, \dots, a_{r-1} can be obtained by repeating the above steps.
- (e) $-a_r$ is a root of indicial equation of $L \otimes (\tau - 1/ct^v(1 + a_1 t^{1/r} + \dots + a_{r-1} t^{(r-1)/r}))$.

The mathematics of the above procedure can be found in [6], [9] and [26].

Remark The algorithm is similar to computing Puiseux expansion for $L \in \mathbb{C}((t))[y]$. In Step (b), we let $L_{cv} := L \otimes (\tau - 1/(ct^v))$ to divide ct^v away in g , so that L_{cv} has right hand factor of $\tau - (1 + a_1 t^{1/r} + \dots + a_r t^{r/r})$. While computing Puiseux expansion, we substitute $y \rightarrow y + ct^v$ to subtract the term we have found, see [26, Theorem 3.1].

Remark Before we compute generalized exponents, we don't know the ramification r . So before we compute Newton Δ -polygon in Step b, we compute indicial equation first and check whether it is polynomial in terms of n . If so, root of indicial equation will be a_n and $n = 1$. If not, we compute Newton Δ -polygon and take the slope that is between $(0, 1]$.

Example 3.2.18. First we will see a difference operator

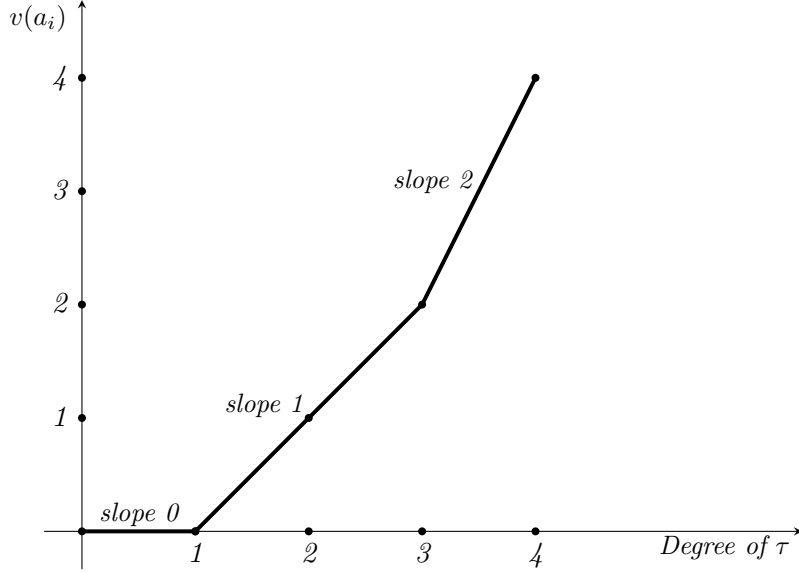
$$L_{ex} = \tau^4 - (x^2 + 5x + 9)\tau^3 + (2x^3 + 11x^2 + 21x + 14)\tau^2 + (-x^4 - 4x^3 - 4x^2 + x + 2)\tau - (x+1)^2(x+2)x.$$

Since order of $L = |\text{genexp}(L)|$ for a difference operator L , we need to find 4 generalized exponent of L_{ex} . To compute Newton τ -polygon of L_{ex} we substitute $x = \frac{1}{t}$ in L_{ex} and multiply t^4 to obtain

$$L_t = \tau^4 t^4 + (-9t^4 - 5t^3 - t^2)\tau^3 + (14t^4 + 21t^3 + 11t^2 + 2t)\tau^2 + (2t^4 + t^3 - 4t^2 - 4t - 1)\tau - 2t^3 - 5t^2 - 4t - 1$$

so that $L_t \in \mathbb{C}[t, \tau]$.

Then Newton τ -polygon of L_t is the following graph.



$$M_2(L_t) = \{(3, 2), (4, 4)\}, \quad Np_2(L_t) = X - 1$$

$$M_1(L_t) = \{(1, 0), (2, 1), (3, 2)\}, \quad Np_1(L_t) = -X^2 + 2X - 1 = -(X - 1)^2$$

$$M_0(L_t) = \{(0, 0), (1, 0)\}, \quad Np_0(L_t) = -X - 1$$

At slope 2, we will get a generalized exponent in a form of $t^{-2}(1 + \dots)$. By computing indicial equation of $L_t \otimes (\tau - t^2)$, we get $3 - n$. Since it is a polynomial in terms of n , we get a generalized exponent $t^{-2}(1 + 3)$.

At slope 1, we will get generalized exponents in a form of $t^{-1}(1 + \dots)$. The fact that $Np_1(L_t)$ has a root at 1 with multiplicity of 2 implies that we will get 2 generalized exponent. The indicial equation of $Ld := L \otimes (\tau - t)$ is 1. Computing Newton Delta-polygon of Ld returns $\{[-1, -1 + X], [0, -1 - X], [1/2, -(X - 1)(X + 1)]\}$, but we only take slope that is between (0, 1]. The indicial equation of both $L_t \otimes (\tau - t(1 + t^{\frac{1}{2}}))$ and $L_t \otimes (\tau - t(1 - t^{\frac{1}{2}}))$ are $\frac{3}{2} + 2n$ hence $n = -\frac{3}{4}$. Thus we get two generalized exponents $t^{-1}(1 + t^{\frac{1}{2}} + \frac{3}{4}t)$ and $t^{-1}(1 - t^{\frac{1}{2}} + \frac{3}{4}t)$.

At slope 0, Indicial equation of $L \otimes (\tau - (-1))$ is $2 - n$. Thus, we get a generalized exponent $-t^0(1 + 2t)$.

In all,

$$\text{genexp}(L_{ex}) = \left\{ t^{-2}(1-3t), -(1-2t), t^{-1}\left(1+t^{\frac{1}{2}}+\frac{3}{4}t\right), t^{-1}\left(1-t^{\frac{1}{2}}+\frac{3}{4}t\right) \right\}$$

Remark In the implementation it is more convenient to compute integer powers than fractional powers. Therefore, we replace t with t_r where t_r is a new variable that represents $t^{1/r}$. When computing $\tau(t_r)$ we take $nr+2$ term of the equation 3.4 where n is the order of the difference operator.

Remark $L_{ex} = MN$ where $M = \tau^2 - (x+1)(x+2)\tau - (x+1)(x+2)$ and $N = \tau^2 - (2x+3)\tau + x(x+1)$. The generalized exponents of M are $\{t^{-2}(1+t), -1\}$ and the generalized exponents of N are $\{t^{-1}(1+t^{\frac{1}{2}}+\frac{3}{4}t), t^{-1}(1-t^{\frac{1}{2}}+\frac{3}{4}t)\}$. A generalized exponent of M , $t^{-2}(1+t)$ corresponds to a generalized exponent of L_{ex} , $t^{-2}(1-3t)$. The coefficient of t inside the parenthesis, 1 and -3, has been obtained from the indicial equation. We can see that these coefficient has been shifted by an integer by Lemma 3.2.5. This is same to a generalized exponent of M , -1 and a generalized exponent of L_{ex} , $-(1-2t)$.

Example 3.2.19. Next lets consider the difference operator

$$L_{WM} = \tau^2(2n+2\nu+3+2x) + (2z-4\nu-4x-4)\tau - 2n+1+2\nu+2x$$

which is an operator from the table in Section 5.2. Suppose

$$\text{genexp}(L_{WM}) = \{c_1 t_1^v(1+a_1 t^{\frac{1}{2}}+a_2 t), c_2 t_2^v(1+b_1 t^{\frac{1}{2}}+b_2 t)\}.$$

The slope of the Newton τ -polygon of L_{WM} is 0 and the corresponding Newton τ -polynomial is $2(t-1)^2$. So, $c_1 = v_1 = c_2 = v_2 = 1$. Since $\text{Ind}_{L_{WM}} = 2z$, $\text{Ind}_{L_{WM}}$ has no root, that is, a_1 and b_1 are not 0. So we need to calculate the Newton Δ -polygon of L_{WM} . Then the slope of the Newton δ -polygon is $\frac{1}{2}$ and its Newton δ -polynomial is $2z+2t^2$, which gives $a_1 = \sqrt{-z}$ and $b_1 = -\sqrt{-z}$. The indicial equation of both $L_{WM} \otimes (\tau - 1/(\sqrt{-z}t^{\frac{1}{2}}))$ and $L_{WM} \otimes (\tau - (1/\sqrt{-z}t^{\frac{1}{2}}))$ is $-64z(1+2z+4\mu) + 256nz$, so the root of the indicial equation is $n = \frac{1}{4} + \frac{1}{2}z + \mu$. Thus, $a_2 = b_2 = -n$,

$$\text{genexp}(L_{WM}) = \left\{ 1 + \sqrt{-z}t^{\frac{1}{2}} - \left(\frac{1}{4} + \frac{1}{2}z + \mu\right)t, 1 - \sqrt{-z}t^{\frac{1}{2}} - \left(\frac{1}{4} + \frac{1}{2}z + \mu\right)t \right\}, \text{ and}$$

$$\text{Gquo} = \{1 - 2\sqrt{-z}t^{\frac{1}{2}} - 2zt, 1 + 2\sqrt{-z}t^{\frac{1}{2}} - 2zt\}.$$

CHAPTER 4

LIOUVILLIAN SOLUTIONS

A function $u(x)$ is called hypergeometric if $\frac{u(x+1)}{u(x)}$ is a rational function. Examples of such functions are Catalan numbers $C(x) = \frac{1}{x+1} \binom{2x}{x}$, $\Gamma(x)$ or 2^x . Sequence A000246=(1, 1, 1, 3, 9, 45, 225, 1575, 11025, 99225, ...) in [1] represents “Number of permutations in the symmetric group S_n that have odd order.” It satisfies recurrence relation $u(x+2) = u(x+1) + x(x+1)u(x)$ for $x \geq 2$, which corresponds to the difference operator $\tau^2 - \tau - x(x+1)$. Giving this operator, and the initial condition, to *solver* produces:

$$v(1) = 1, \tag{4.1}$$

$$v(x) = \frac{2^{x+1}}{2\pi(x-1)} \Gamma\left(\frac{1}{2}x + \frac{1}{2}\right)^2 \quad \text{when } x \text{ is even,} \tag{4.2}$$

$$v(x) = \frac{2^x}{2\pi} \Gamma\left(\frac{1}{2}x\right)^2 \quad \text{when } x \text{ is odd } \geq 3. \tag{4.3}$$

If we substitute x with $2x$ to both equations (4.2) and (4.3), they become hypergeometric terms. One hypergeometric term describes $(v(1), v(3), v(5), \dots)$ and the other describes $(v(2), v(4), v(6), \dots)$. So, $(v(1), v(2), v(3), \dots)$ is an *interlacing* of two hypergeometric terms. Such sequences are called *Liouvillian*, see [18, Definition 3.3] for a formal definition. Lemma 4.2.1 says if an irreducible operator has Liouvillian solution then it is gauge equivalent to an operator of the form $\tau^n + c\phi(x)$ where $\phi \in \mathbb{C}(x)$ is a monic rational function¹ and $c \in \mathbb{C}$.

First base equation we will introduce from the table is $\tau^n + c\phi(x) \in D$. This equation is parameterized with $c\phi(x) \in \mathbb{C}(x)$. In this chapter we use valuation growth to construct $c\phi(x)$ so that the input operator L_{inp} is gauge equivalent an operator of the form $\tau^n + c\phi(x) \in D$ (details are provided only for $n = 2$ and $n = 3$, but it is easy to generalize to higher order). Since term transformation is multiplying by a hypergeometric solution to a solution of another operator, we only need gauge transformation for difference equations with Liouvillian solutions. The main part of this chapter is to solve this problem with Theorems 4.2.5 and 4.2.7.

4.1 Definitions

Definition 4.1.1. We say $a, b \in \mathbb{C}(x)$ are n -equivalent if $\frac{a}{b} = \frac{\tau^n(r)}{r}$ for some non-zero $r \in \mathbb{C}(x)$ and denote $a \sim_n b$.

¹see Definition 4.1.2 for the definition of monic rational function

Note that n -equivalence is similar to [16, Section 3.2.1] (the new results in this section are found after the Problem Statement below).

Definition 4.1.2. A rational function is said to be monic if it is a quotient of monic polynomials. We write a non-zero rational function as $c\phi(x)$ where $\phi(x)$ is a monic rational function and $c \in C^*$.

Example 4.1.3. If $\phi(x) = (x - q_1)^{n_1} R(x)$ and $\tilde{\phi}(x) = (x - q_1 - n)^{n_1} R(x)$ then $\phi, \tilde{\phi}$ are n -equivalent. This means that up to n -equivalence one can shift roots or poles by multiples of n . If all roots and poles of $c\phi(x)$ are in \mathbb{Z} then $c\phi(x)$ is 1-equivalent to a function of the form cx^n and 3-equivalent to a function of the form $cx^{n_1}(x-1)^{n_2}(x-2)^{n_3}$ for some $n_1, n_2, n_3 \in \mathbb{Z}$ with $n = n_1 + n_2 + n_3$.

We will denote $\det(L)$ as the determinant of the companion matrix of L . Let $L = a_n\tau^n + \dots + a_0\tau^0$ then $\det(L) = (-1)^n \frac{a_0}{a_n}$. If $L \sim_g M$ and G is gauge transformation from L to M then $\det(M) = \frac{\tau(\det(A_G))}{\det(A_G)} \det(L)$ by Equation (2.3). Thus,

$$\det(L) \sim_1 \det(M). \quad (4.4)$$

If $M = \tau^n + c\phi$ with ϕ monic then Equation (4.4) implies

$$\frac{a_0}{a_n} \sim_1 c\phi \text{ and } c = \text{lc}(a_0)/\text{lc}(a_n) \quad (4.5)$$

where $\text{lc}(a_i)$ denotes the leading coefficient of a_i . Note that this is similar to the proof of Lemma 3.13 in [16].

4.2 Main Theorem

Lemma 4.2.1. [18, Lemma 4.1],[16, Prop. 3.1], [10, Prop. 55] If $L = a_n\tau^n + \dots + a_0\tau^0$ is irreducible then there exist Liouvillean Solutions if and only if there exists $c\phi(x) \in \mathbb{C}(x)$ such that

$$a_n\tau^n + \dots + a_0\tau^0 \sim_g \tau^n + c\phi(x)$$

Lemma 4.2.2. If $\phi(x) \sim_n \tilde{\phi}(x)$ and if L is gauge equivalent to $\tau^n + c\phi$ then L is also gauge equivalent to $\tau^n + c\tilde{\phi}(x)$.

Proof. $\tilde{\phi}/\phi = \tau^n(r)/r$ for some $r \in C(x)$ by definition of n -equivalence. Then $M := \tau^n + c\phi$ and $\tilde{M} := \tau^n + c\tilde{\phi}$ are gauge equivalent because multiplying by r is a bijection from $V(M)$ to $V(\tilde{M})$. Since gauge equivalence is an equivalence relation, $L \sim_g \tilde{M}$. \square

Problem Statement

Operators of the form $\tau^n + c\phi$ can be solved easily (see subsection 4.2.1 for details). If L is gauge equivalent to an operator of the form $M = \tau^n + c\phi$ then we can solve L as well. However, given only L , not M , we need to find $c\phi$ up to n -equivalence (see Lemma 4.2.2) but a_0/a_n only provides it up to 1-equivalence.

Lemma 4.2.3. Let $M = \tau^3 + cx^{n_1}(x-1)^{n_2}(x-2)^{n_3}$ for some $c \in C^*$ and let $p = \mathbb{Z} \in \overline{\mathbb{C}}/\mathbb{Z}$ then $\min(\bar{g}_p(M)) = \min\{n_1, n_2, n_3\}$ and $\max(\bar{g}_p(M)) = \max\{n_1, n_2, n_3\}$.

Proof. By Lemma 3.1.5,

$$v_{\varepsilon,l}(\tilde{u}) = \min\{v_{\varepsilon}(\tilde{u}(-3)), v_{\varepsilon}(\tilde{u}(-2)), v_{\varepsilon}(\tilde{u}(-1))\}$$

and

$$v_{\varepsilon,r}(\tilde{u}) = \min\{v_{\varepsilon}(\tilde{u}(3)), v_{\varepsilon}(\tilde{u}(4)), v_{\varepsilon}(\tilde{u}(5))\}$$

for all non-zero $\tilde{u} \in V_p(M_{\varepsilon})$. Now

$$v_{\varepsilon}(\tilde{u}(3)) = v_{\varepsilon}(\tilde{u}(-3)) + n_1$$

$$v_{\varepsilon}(\tilde{u}(4)) = v_{\varepsilon}(\tilde{u}(-2)) + n_2$$

$$v_{\varepsilon}(\tilde{u}(5)) = v_{\varepsilon}(\tilde{u}(-1)) + n_3.$$

The values of $v_{\varepsilon}(\tilde{u}(-3)), v_{\varepsilon}(\tilde{u}(-2)), v_{\varepsilon}(\tilde{u}(-1))$ in $\mathbb{Z} \cup \{\infty\}$ can be chosen arbitrarily by choosing suitable $\tilde{u} \in V_p(M_{\varepsilon})$. Doing so it is easy to check from the five equations above that the smallest resp. largest possible value one can obtain for

$$g_{p,\varepsilon}(\tilde{u}) = v_{\varepsilon,r}(\tilde{u}) - v_{\varepsilon,l}(\tilde{u})$$

is $\min\{n_1, n_2, n_3\}$ resp. $\max\{n_1, n_2, n_3\}$. So $\min(\bar{g}_p(M)) = \min\{n_1, n_2, n_3\}$ and $\max(\bar{g}_p(M)) = \max\{n_1, n_2, n_3\}$. \square

Definition 4.2.4. Let $L = a_n\tau^n + \dots + a_0$ and let p_1, \dots, p_k be the finite singularities of L . Write $p_i = q_i + \mathbb{Z}$ for some $q_i \in \bar{C}$ then $a_0/a_n \sim_1 c \prod_{i=1}^k (x - q_i)^{n_i}$ for some $c \in C$ and $n_i \in \mathbb{Z}$. We call this n_i the \sim_1 -exponent of L at p_i .

We will first sketch the key idea before giving the Theorem below. Suppose that the operator L is gauge equivalent to some unknown $M = \tau^3 + c\phi$, and suppose for example that M is as in Lemma 4.2.3. The \sim_1 -exponent of M (and hence of L by equation (4.4)) at p is $n_1 + n_2 + n_3$. Our strategy is now this: to find M , our algorithm needs to compute numbers n_1, n_2, n_3 at every singularity p . It is easy to compute the sum of these three numbers by taking the \sim_1 -exponent of L . But we can also compute the minimum and the maximum of these three numbers using Lemma 4.2.3 combined with Theorem 3.1.9. Knowing the minimum, maximum, and sum, of three numbers, that determines those numbers up to a permutation. That is the key idea of our algorithm for order 3, and in the Theorem below.

Theorem 4.2.5. Let $L = a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0$ where $a_i \in C[x]$. Let $\{p_1, \dots, p_k\} \subseteq \bar{C}/\mathbb{Z}$ be the set of finite singularities of L . Write $p_i = q_i + \mathbb{Z}$ for some $q_i \in \bar{C}$. Let $M_i = \max(\bar{g}_{p_i}(L))$, $m_i = \min(\bar{g}_{p_i}(L))$ and $e_i = n_i - M_i - m_i$ where n_i is the \sim_1 -exponent of L at p_i . If L is gauge equivalent to an operator of the form $\tau^3 + c\phi$ for some monic rational function $\phi \in C(x)$ and $c \in C^*$ (c is given in equation (4.5)) then

$$\phi \sim_3 \prod_i^k (x - q_i)^{n_{i,1}} (x - (q_i + 1))^{n_{i,2}} (x - (q_i + 2))^{n_{i,3}}$$

where $(n_{i,1}, n_{i,2}, n_{i,3})$ is a permutation of (M_i, m_i, e_i) .

Proof. Let $M = \tau^3 + c\phi$ and gauge equivalent to L . We may assume that the singularities of M are a subset of $\{p_1, \dots, p_k, p_{k+1}, \dots, p_l\}$ for some $p_{k+1}, \dots, p_l \in \overline{C}/\mathbb{Z}$. Write $p_i = q_i + \mathbb{Z}$ for some $q_i \in \overline{C}$. Now,

$$\phi \sim_3 \prod_{i=1}^l (x - q_i)^{n_{i,1}} (x - (q_i + 1))^{n_{i,2}} (x - (q_i + 2))^{n_{i,3}} \quad (4.6)$$

for some $n_{i,1}, n_{i,2}$ and $n_{i,3}$ as explained in Example 4.1.3. Then

$$\frac{a_0}{a_3} \sim_1 c\phi \sim_1 c \prod_{i=1}^l (x - q_i)^{n_i},$$

see Equation (4.5) and Example 4.1.3, with

$$n_i = n_{i,1} + n_{i,2} + n_{i,3}. \quad (4.7)$$

By Theorem 3.1.9 and Lemma 4.2.3,

$$M_i = \max(\overline{g}_{p_i}(L)) = \max(\overline{g}_{p_i}(\tau^3 + c\phi)) = \max\{n_{i,1}, n_{i,2}, n_{i,3}\} \quad m_i = \min(\overline{g}_{p_i}(L)) = \min(\overline{g}_{p_i}(\tau^3 + c\phi)) = \min\{n_{i,1}, n_{i,2}, n_{i,3}\}$$

For $i > k$ we have $\overline{g}_{p_i}(L) = \{0\}$ by Lemma 3.1.8 (p_i is not a singularity of L if $i > k$) and so $\max\{n_{i,1}, n_{i,2}, n_{i,3}\} = 0$ and $\min\{n_{i,1}, n_{i,2}, n_{i,3}\} = 0$. In all, $n_{i,1} = n_{i,2} = n_{i,3} = 0$ for all $i > k$. Thus, we can replace l in equation (4.6) by k :

$$\phi \sim_3 \prod_i^k (x - q_i)^{n_{i,1}} (x - (q_i + 1))^{n_{i,2}} (x - (q_i + 2))^{n_{i,3}}.$$

The maximum of $n_{i,1}, n_{i,2}, n_{i,3}$ is M_i and the minimum is m_i , and so the remaining number must be $e_i := n_i - M_i - m_i$ by Equation (4.7). This determines $n_{i,1}, n_{i,2}, n_{i,3}$ up to a permutation. \square

Remark. If $p \in \overline{C}/\mathbb{Z}$ is a apparent singularity (Definition 3.1.7) of L then $n_1 = n_2 = n_3 = 0$ so then p will not be a singularity of $\tau^3 + c\phi$. Hence such p are not needed for constructing ϕ in our algorithm.

Example 4.2.6. Suppose $L = a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0$ and $p = \mathbb{Z} \in \overline{C}/\mathbb{Z}$ is the only singularity of L . If $L \sim_g (\tau^3 + c\phi)$ for some monic rational function $\phi(x) \in C(x)$ and $c \in C^*$ then for some integers n_1, n_2, n_3 one has $cx^{n_1}(x-1)^{n_2}(x-2)^{n_3} \sim_3 c\phi(x) \sim_1 \frac{a_0}{a_3} \sim_1 cx^n$, where $n = n_1 + n_2 + n_3$. Let $M = \max(\overline{g}_p(L))$, $m = \min(\overline{g}_p(L))$ and $e = n - M - m$. Then the ordered triple (n_1, n_2, n_3) is a permutation of M, m and e . If M, m and e are all distinct numbers this leaves $3! = 6$ possibilities for the ordered triple (n_1, n_2, n_3) .

More generally, if there are k singularities then we have $\leq 6^k$ combinations, with equality when M_i, m_i and e_i are all distinct for each singularity.

Theorem 4.2.7. *Let $L = a_2\tau^2 + a_1\tau + a_0$ where $a_i \in C[x]$. Suppose the singularities are $\{p_1, \dots, p_k\} \subseteq \overline{C}/\mathbb{Z}$. Write $p_i = q_i + \mathbb{Z}$ for some $q_i \in \overline{C}$. Let $M_i = \max(\overline{g}_{p_i}(L))$ and $m_i = \min(\overline{g}_{p_i}(L))$. If L is gauge equivalent to operator of the form $\tau^2 + c\phi$ then*

$$\phi \sim_2 \prod_i \begin{cases} (x - q_i)^{M_i}(x - (q_i + 1))^{m_i} \\ \text{or} \\ (x - q_i)^{m_i}(x - (q_i + 1))^{M_i} \end{cases}$$

and c is as in Equation (4.5).

The proof is similar to the proof of Theorem 4.2.5. As an example, if $p = \mathbb{Z}$ is the only singularity of L then $\phi \sim_2 x^M(x-1)^m$ or $x^m(x-1)^M$. More generally, the number of combinations that the algorithm need to check is 2^l where l is the number of p_i for which $M_i \neq m_i$. For each combination we find a candidate ϕ up to 2-equivalence.

4.2.1 Solutions of $\tau^n + c\phi$

Solutions of $L = \tau^n + c\phi$ can be found easily. First find a solution $u(x)$ of $\tau + c\phi(nx)$. Let $v(x) = u(x/n)$ then

$$\begin{aligned} v(x+n) &= u((x+n)/n) = u(x/n + 1) \\ &= -c\phi(nx/n)u(x/n) \\ &= -c\phi(x)v(x). \end{aligned}$$

Thus $v(x)$ is a solution of $L = \tau^n + c\phi(x)$, and $(\xi)^x v(x)$ is also solution of L for any $\xi \in \overline{C}$ with $\xi^n = 1$. We obtain a basis of $V(L)$ this way.

4.3 Algorithms for order 2 and 3

Let C be a field of characteristic zero. Given $L = a_2\tau^2 + a_1\tau + a_0 \in C(x)[\tau]$, after clearing denominators we may assume that $a_0, a_1, a_2 \in C[x]$. Algorithm Tausqsols resp. Taucbsols below uses Theorem 4.2.7 resp. 4.2.5 to compute a set comb, the set of all candidates for ϕ . It then checks each $\phi \in \text{comb}$.

Note that the two algorithms below only search for ϕ defined over the field C , and that C must be given in the input. If there exist Liouvillian solutions with ϕ defined not over C but over some algebraic extension C' of C , then in order to find these solutions, we need to call the algorithm with C' instead of C in the input. The problem of finding these field extensions C' of C has already been solved for hypergeometric solutions in [15, Section 8] and the same approach works here as well. The only difference is that here we have additional information that can be used to further reduce the search for C' , for instance, unlike for hypergeometric solutions, in our situation the minimal field extension needed to find ϕ must necessarily be Galois over C with cyclic Galois group (this restriction is of course only useful for order > 2 because an extension of degree 2 is always cyclic). Our implementation for order 2 uses the same approach as in [15] to determine the fields C' for which we have to call Tausqsols(C', L) in order to find all Liouvillian solutions.

Algorithm 4.3.1. *Tausqsols*

Input: A field C of characteristic 0, and an operator $L = a_2\tau^2 + a_1\tau + a_0$ with $a_0, a_1, a_2 \in C[x]$ and $a_2 \neq 0, a_0 \neq 0$.

Output: A basis of solutions of L if there exists an operator of the form $\tau^2 + c\phi \in C(x)[\tau]$ that is gauge equivalent to L . Otherwise the empty set.

1. Let S be the irreducible factors of a_2a_0 over C up to 1-equivalence.
2. $c := \text{lc}(a_0)/\text{lc}(a_2)$ as in equation (4.5).
3. $\text{comb} := \{1\}$.
4. For $s \in S$ do
 - (a) $p := a$ root of s .
 - (b) $m := \min(\bar{g}_p(L)), M := \max(\bar{g}_p(L))$.
 - (c) $T := \{s(x)^m s(x-1)^M, s(x)^M s(x-1)^m\}$.
 - (d) $\text{comb} := \{ij \mid i \in \text{comb}, j \in T\}$.
5. For each $\phi \in \text{comb}$, check if there exists a gauge transformation from $\tau^2 + c\phi$ to L , and if so, then
 - (a) Compute a basis of solutions of $\tau^2 + c\phi$.
 - (b) Apply the gauge transformation to the solutions of $\tau^2 + c\phi$.
 - (c) Return the result of step 5b as output and stop the algorithm.
6. Return \emptyset .

Algorithm 4.3.2. *Taucbsols*

Input: A field C of characteristic 0, and an $L = a_3\tau^3 + a_2\tau^2 + a_1\tau + a_0$ with $a_0, a_1, a_2, a_3 \in C[x]$ and $a_3 \neq 0, a_0 \neq 0$.

Output: A basis of solutions of L if there exists an operator of the form $\tau^3 + c\phi \in C(x)[\tau]$ that is gauge equivalent to L . Otherwise the empty set.

1. Let S be the irreducible factors of a_3a_0 over C up to 1-equivalence.
2. $c := \text{lc}(a_0)/\text{lc}(a_3)$ as in equation (4.5).
3. We can write

$$\frac{a_0}{a_3} = c \prod_{\substack{s \in S \\ i \in \mathbb{Z}}} s(x-i)^{n_{i,s}}$$

with only finitely many $n_{i,s} \neq 0$. Then for each $s \in S$ let $l_s := \sum_i n_{i,s} \in \mathbb{Z}$.

4. $\text{comb} := \{1\}$.
5. For $s \in S$ do
 - (a) $p :=$ root of s .

(b) $m := \min(\bar{g}_p(L))$, $M := \max(\bar{g}_p(L))$, $e := l_s - M - m$.

(c) $E :=$ the set of all permutations of $[m, M, e]$.

(d) $T := \{s(x)^i s(x-1)^j s(x-2)^k \mid [i, j, k] \in E\}$.

(e) $\text{comb} := \{ij \mid i \in \text{comb}, j \in T\}$.

6. For each $\phi \in \text{comb}$, check if there exists a gauge transformation from $\tau^3 + c\phi$ to L , and if so, then

(a) Compute a basis of solutions of $\tau^3 + c\phi$.

(b) Apply the gauge transformation to the solutions of $\tau^3 + c\phi$.

(c) Return the result of step 6b as output and stop the algorithm.

7. Return \emptyset .

See Section 4 of [15] for computing the set of valuation growths of a difference operator (an implementation is available in Maple as the undocumented command ‘LREtools/g_p’).

4.4 Example

We will follow Algorithm Tausqsols with the operator $L =$

$$(3 + 2x)(x + 4)(x + 3)\tau^2 - (8x^2 + 32x + 36)\tau - 16x(2x + 5)(x + 1).$$

First we get $S = \{x, x - 1/2\}$. Let $p = 0 + \mathbb{Z}$ and $p' = 1/2 + \mathbb{Z}$, then

$$\bar{g}_p(L) = \{-2, -1, 0, 1, 2\} \quad \text{and} \quad \bar{g}_{p'}(L) = \{0\}.$$

So p' is apparent singularity of L and it has no role in constructing ϕ . Thus

$$\text{comb} = \left\{ \frac{(x-1)^2}{x^2}, \frac{x^2}{(x-1)^2} \right\}$$

and

$$c = \frac{\text{lc}(a_0(x))}{\text{lc}(a_2(x))} = \frac{\text{lc}(-16x(2x+5)(x+1))}{\text{lc}((3+2x)(x+4)(x+3))} = -16.$$

So we have two candidates $\tau^2 - 16\frac{(x-1)^2}{x^2}$ and $\tau^2 - 16\frac{x^2}{(x-1)^2}$, and the algorithm checks if any of these candidates is gauge equivalent to L . It finds that $\tau^2 - 16\frac{x^2}{(x-1)^2}$ is gauge equivalent to L and finds the gauge transformation

$$g_1(x)\tau + g_0(x) = \frac{1}{x^3 + 2x^2}\tau + \frac{4x}{(x^2 - 1)^2}. \quad (4.8)$$

Using Section 4.2.1 we get a basis of solutions of $\tau^2 - 16\frac{(x+1)^2}{x^2}$, namely

$$v(x) \text{ and } (-1)^x v(x), \text{ where } v(x) = \frac{16^{\frac{x}{2}} \Gamma(\frac{x}{2} + \frac{1}{2})^2}{\Gamma(\frac{x}{2})^2}. \quad (4.9)$$

By applying the gauge transformation (4.8) to the solution (4.9) we get

$$\begin{aligned} &g_1(x)v(x+1) + g_0(x)v(x), \\ &(-1)^{x+1}g_1(x)v(x+1) + (-1)^xg_0(x)v(x) \end{aligned}$$

as a basis of solutions of L , where $g_1(x), g_0(x)$ are given in equation (4.8).

The algorithm presented in [18] would construct an operator \tilde{L} and then compute its hypergeometric solutions. In the example L given above, we find (we used Khmelnov's [20] Maple implementation to compute \tilde{L})

$$\begin{aligned} \tilde{L} = &(x+3)(x+2)(8x^2+16x+9)(5+2x)^2\tau^2 \\ &+ (-8100 - 35904x - 1024x^6 - 9216x^5 \\ &- 66112x^2 - 63744x^3 - 33664x^4)\tau \\ &+ 256x(x+1)(8x^2+32x+33)(1+2x)^2. \end{aligned}$$

Apparent singularities of L can become non-singular in the operator \tilde{L} , and non-singular points can become apparent singularities, but this does not matter because neither apparent singularities nor non-singular points contribute to the combinatorial problem.

Concerning the singularities that do contribute to the combinatorial problem, each singularity $p = q + \mathbb{Z}$ of L corresponds to $n := \text{ord}(L)$ singularities of \tilde{L} , namely $p_1 = q/n + \mathbb{Z}$, $p_2 = (q+1)/n + \mathbb{Z}$, $p_3 = (q+2)/n + \mathbb{Z}, \dots, p_n = (q+n-1)/n + \mathbb{Z}$. The set $\bar{g}_p(L)$ at the singularity p of L is the same as the set $\bar{g}_{p_i}(\tilde{L})$ at each of the n singularities p_1, \dots, p_n of \tilde{L} .

So in this example, singularity $p = 0 + \mathbb{Z}$ of L corresponds to two singularities $p_1 = 0 + \mathbb{Z}$ and $p_2 = 1/2 + \mathbb{Z}$ of \tilde{L} , each of which has the same set of valuation growths $\{-2, -1, 0, 1, 2\}$ as L has at p . We verified with a Maple computation that $\bar{g}_{p_i}(\tilde{L})$, for $i = 1, 2$, is indeed equal to $\{-2, -1, 0, 1, 2\}$. Note that \tilde{L} has another singularity, given by a root of $8x^2 + 16x + 9$ (or of $8x^2 + 32x + 33$ which is 1-equivalent to it). Since this singularity corresponds to a regular point of L , we conclude that this must be an apparent singularity, and indeed, we verified with a Maple computation that the set of valuation growths is $\{0\}$ at this singularity.

If we solve \tilde{L} with the algorithm *hypergeomsols* in Maple, then it has to choose an element of $\bar{g}_{p_1}(\tilde{L})$ and an element of $\bar{g}_{p_2}(\tilde{L})$, and there are $5 \times 5 = 25$ ways to make such choices. Thus, the number of combinations coming from the finite singularities is 25. In contrast, our algorithm had only 2 combinations to check. For order 3, the algorithm in [18] calls *hypergeomsols* several times (see [2] to reduce the number of such calls) and if N is the number of combinations that *hypergeomsols* has to check in one such call, then the number of combinations in our algorithm is at most N^r where $r = \max\{\log(3!)/\log(3^3), \log(3)/\log(2^3)\} = 0.54\dots$. So we have reduced the combinatorial problem by roughly the square root.

CHAPTER 5

SPECIAL FUNCTIONS

Many special functions satisfy recurrences w.r.t their parameters (see [5]). In this chapter, we used these recurrences to construct a base equations for our table. We will start by explaining with two examples in section 5.1 how *solver* uses the table. In section 5.2, we will give the table of base equations and their local data. This table has been constructed for computational purpose.

In chapter 4, the invariant data was the finite singularities. In this chapter, in addition to finite singularities, we shall also use local data at infinity (generalized exponents). The reason for using both in this chapter is the following: The more solved equations we add to the table, the stronger the solver will become; however, we can only add equations to the table if the parameters in those equations can be computed from the invariant data that we compute. This is why we must have an implementation for all the local data described in section 3.1 and section 3.2

For an input operator L_{inp} , base equation is chosen by comparing degree of T of elements in $Gquo(L_{inp})$, $Deg_T(L_{inp})$, and the set $Val(L_{inp})$ from those of the table. Suppose $Deg_T(L_{inp}) = 3$ and $Val(L_{inp})$ is an empty set. Then it matches with $LbIK$ or $LbYJ$ (see section 5.2). If coefficient of T^2 is positive we use $LbJY$ and use $LbIK$ if negative.

If $Val(L_{inp})$ is not an empty set then $|Val(L_{inp})|$ changes depending on values of parameters. Legendre function is a special case of Jacobi polynomial ($P_x(z) = P_x^{0,0}(z)$)¹. If a, b in Ljc are both integers, then $Val(Ljc) = \{[0, 2]\}$ and $Deg_T(Ljc) = 0$. These data coincide those of Lgd 's. In this case we construct base equation with Lgd first since it has less parameters to find.

Deg_T can vary depending on parameters also. If $c = 0$ in Lkm then $Deg_T(Lkm) = 0$. In this case, *solver* compares data of Val first to find right base equation.

5.1 Examples

Example 5.1.1. Sequence $A096121 = (2, 8, 60, 816, 17520, 550080, \dots)$ in [1] represents “Number of full spectrum rook’s walks on a $(2 \times n)$ board” and it is a solution of the recurrence operator $A = \tau^2 - (x + 1)(x + 2)\tau - (x + 1)(x + 2)$. The local data of A are

$$Gquo(A) = \{-t^2(1 - t)\} \quad \text{and} \quad Val = \{ \}.$$

¹see section 5.2 for the notation

The local data of A matches the operator L_{IK} in the table in Section 5.2. Before we can call algorithm Find GT-transformation (see Algorithm 2.3.11) we need to find explicit values for the unknown constants z and ν appearing in L_{IK} . Since $\tau(I_{\nu+x}(z)) = I_{\nu+x+1}(z)$ and τ is a gauge transformation, we only need $\nu \bmod \mathbb{Z}$. Comparing $\text{Gquo}(A)$ with $\text{Gquo}(L_{IK})$ (see table in Section 5.2) using Theorem 3.2.14 gives $-1 \equiv -1 - 2\nu \bmod \mathbb{Z}$ and $-\frac{z^2}{4} = -1$. Hence $\nu \in \frac{1}{2} + \mathbb{Z}$ or $\nu \in 0 + \mathbb{Z}$ and $z = \pm 2$. So, if A can be reduced to L_{IK} for some parameter value, then A can be reduced to one of:

$$\begin{aligned} & -2\tau^2 + (2 + 2x)\tau + 2, \quad -2\tau^2 + (3x + 1)\tau + 2, \\ & 2\tau^2 + (2 + 2x)\tau - 2, \quad 2\tau^2 + (3x + 1)\tau - 2. \end{aligned}$$

(These are L_{IK} with $\nu \in \{0, \frac{1}{2}\}$, $z \in \{2, -1\}$.) Then algorithm Find GT-transformation finds that A can be reduced to $-2\tau^2 + (2 + 2x)\tau + 2$. It also finds the gauge transformation 1 and the term product $\tau - (x + 1)$. From the list, a basis of solutions of $-2\tau^2 + (2 + 2x)\tau + 2$ is

$$\{I_x(-2), K_x(2)\}.$$

By applying the gauge transformation and the term product we get a basis of solutions of A as

$$\{I_x(-2)\Gamma(x + 1), K_x(2)\Gamma(x + 1)\}.$$

Example 5.1.2. Sequence $A005572 = (1, 4, 17, 76, 354, 1704, 8421, \dots)$ in [1] represents “Number of walks on cubic lattice starting and finishing on the xy -plane and never going below it” and it is a solution of the recurrence operator $H = (x + 4)\tau^2 + (-20 - 8x)\tau + (12x + 12)$. This same example has been used in [17] also. Local data of H are

$$\text{Gquo}(A) = \{\frac{1}{3}, 3\} \quad \text{and} \quad \text{Val} = \{[0, 2]\}.$$

Local data of H matches with the operator L_{2F_1} in the table in Section 5.2. Since $\text{Val} = \{[0, 2]\}$, we get $a, c \in 0 + \mathbb{Z}$. We may take $a = 1$ and $c = 1$ so that ${}_2F_1$ is defined. Comparing Gquo gives $c - 2b \equiv 0 \bmod \mathbb{Z}$ and $1 - z \in \{1/3, 3\}$. By Lemma 4.3 in [17] we need $b \bmod \mathbb{Z}$, so $b \in 0 + \mathbb{Z}$ or $b \in \frac{1}{2} + \mathbb{Z}$. So, if H can be reduced to L_{2F_1} for some parameter values, then H can be reduced to one of:

$$\begin{aligned} & -3(2 + x)\tau^2 + (7 + 4x)\tau - 1 - x, \quad -3(2 + x)\tau^2 + (6 + 4x)\tau - 1 - x \\ & -\frac{1}{3}(2 + x)\tau^2 + (\frac{5}{3} + \frac{4}{3}x)\tau - 1 - x, \quad -\frac{1}{3}(2 + x)\tau^2 + (2 + \frac{4}{3}x)\tau - 1 - x. \end{aligned}$$

(These are L_{2F_1} with $a = 1, b \in \{0, \frac{1}{2}\}, c = 1, z \in \{-2, \frac{2}{3}\}$.) Then algorithm Find GT-transformation finds that H can be reduced to $-\frac{1}{3}(2 + x)\tau^2 + (2 + \frac{4}{3}x)\tau - 1 - x$ with gauge transformation $\frac{1}{x+2}(-2\tau + 3)$ and term product $\tau - 2$. From the table, a solution of $-\frac{1}{3}(2 + x)\tau^2 + (2 + \frac{4}{3}x)\tau - 1 - x$ is

$${}_2F_1(x + 1, \frac{1}{2}; 1; \frac{2}{3}).$$

By applying the gauge transformation and the term product we get a solution of A and after checking initial values, we find that the sequence equals

$$2_x \frac{{}_2F_1(x+1, \frac{1}{2}; 1; \frac{2}{3}) \cdot 3 - {}_2F_1(x+2, \frac{1}{2}; 1; \frac{2}{3}) \cdot 2}{x+2}.$$

5.2 The Table

The following is the table of base equations with their known solutions and local data. Notations of each function in this table are based on [5].

- $LbIK = z\tau^2 + (2 + 2v + 2x)\tau - z$
Solutions: Modified Bessel functions of the first and second kind, $I_{v+x}(z)$ and $K_{v+x}(-z)$
- $LbJY = z\tau^2 - (2 + 2v + 2x)\tau + z$
Solutions: Bessel functions of the first and second kind, $J_{v+x}(z)$ and $Y_{v+x}(z)$
- $LWW = \tau^2 + (z - 2v - 2x - 2)\tau - v - x - \frac{1}{4} - v^2 - 2vx - x^2 + n^2$
Solution: Whittaker function $W_{x,n}(z)$
- $LWM = \tau^2(2n + 2v + 3 + 2x) + (2z - 4v - 4x - 4)\tau - 2n + 1 + 2v + 2x$
Solution: Whittaker function $M_{x,n}(z)$
- $L2F1 = (z - 1)(a + x + 1)\tau^2 + (-z + 2 - za - zx + 2a + 2x + zb - c)\tau - a + c - 1 - x$
Solution: Hypergeometric function ${}_2F_1(a + x, b; c; z)$
- $Ljc = \tau^2 - \frac{1}{2} \frac{(2x+3+a+b)(a^2-b^2+(2x+a+b+2)(2x+4+a+b)z)}{(x+2)(x+2+a+b)(2x+a+b+2)} \tau + \frac{(x+1+a)(x+1+b)(2x+4+a+b)}{(x+2)(x+2+a+b)(2x+a+b+2)}$
Solution: Jacobian polynomial $P_x^{a,b}(z)$
- $Lgd = \tau^2 - \frac{(2x+3)z}{x+2} \tau + \frac{x+1}{x+2}$
Solution: Legendre functions $P_x(z)$ and $Q_x(z)$
- $Lgr = \tau^2 - \frac{2x+3+\alpha-z}{x+2} \tau + \frac{x+1+\alpha}{x+2}$
Solution: Laguerre polynomial $L_x^{(\alpha)}(z)$
- $Lgb = \tau^2 - \frac{2z(m+x+1)}{x+2} \tau - \frac{2m+x}{x+2}$
Solution: Gegenbauer polynomial $C_x^m(z)$
- $Lgr1 = (x + 2)\tau^2 + (x + z - b + 1)\tau + z$
Solution: Laguerre polynomial $L_x^{(b-x)}(z)$
- $Lkm = (a + x + 1)\tau^2 + (-2a - 2x - 2 + b - c)\tau + a + x + 1 - b$
Solution: Kummer's function $M(a + x, b, c)$
- $L2F0 = \tau^2 + (-zb + zx + z + za - 1)\tau + z(b - x - 1)$
Solution: Hypergeometric function ${}_2F_0(a, b - x; ; z)$

- $Lge = (x + 2)\tau^2 + (-ab - d + (a + 1)(1 + x))\tau + ax - a(b + d)$
Solution: Sequences² whose ordinary generating function is $(1 + x)^a(1 + bx)^c$

Operator	Val	Gquo
$LbIK$	$\{\}$	$\{-\frac{1}{4}T^2z^2(1 + (-1 - 2v)T)\}$
$LbJY$	$\{\}$	$\{\frac{1}{4}T^2z^2(1 + (-1 - 2v)T)\}$
LWW	$\{[-n + \frac{1}{2} - v, 1], [n + \frac{1}{2} - v, 1]\}$	$\{-3 - 2\sqrt{2}(1 - \frac{1}{2}\sqrt{2}z)T, -3 + 2\sqrt{2}(1 + \frac{1}{2}\sqrt{2}z)T\}$
LWM	$\{[-n + \frac{1}{2} - v, 1], [n + \frac{1}{2} - v, 1]\}$	$\{1 - 2\sqrt{-z}T - 2zT^2, 1 + 2\sqrt{-z}T - 2zT^2\}$
$L2F1$	$\{[-a + c, 1], [-a, 1]\}$	$\{-\frac{1}{z-1}(1 + (2b - c)T), (-z + 1)(1 + (-2b + c)T)\}$
Ljc	$\{[0, 1], [-a, 1], [-b, 1], [-a - b, 1]\}$	$\{2z^2 - 2z\sqrt{z^2 - 1} - 1, 2z^2 + 2z\sqrt{z^2 - 1} - 1\}$
Lgd	$\{[0, 2]\}$	$\{2z^2 - 2z\sqrt{z^2 - 1} - 1, 2z^2 + 2z\sqrt{z^2 - 1} - 1\}$
Lgr	$\{[-\alpha, 1], [0, 1]\}$	$\{1 + 2\sqrt{-z}T - 2zT^2, 1 - 2\sqrt{-z}T - 2zT^2\}$
$Lgr1$	$\{[0, 1]\}$	$\{zT(1 + 2bT)\}$
Lgb	$\{[-2m, 1]\}$	$\{-2z\sqrt{z^2 + 1} - 2z^2 - 1, 2z\sqrt{z^2 + 1} - 2z^2 - 1\}$
Lkm	$\{[-a, 1], [-a + b, 1]\}$	$\{1 - 2\sqrt{c}T + 2cT^2, 1 + 2\sqrt{c}T + 2cT^2\}$
$L2F0$	$\{[b, 1]\}$	$\{\frac{T}{z}(1 + (b - 2a)T)\}$
Lge	$\{[0, 1], [b + d, 1]\}$	$\{a(1 + (d - b)T), \frac{1}{a}(1 + (-b - d)T)\}$

In case $L2F1$, whenever $b \in [0, -1, -2, \dots]$, ${}_2F_1(a+x, b; c; z)$ satisfies a first order recurrence equation as mentioned in [17, Remark 4.1]. So, this case is not of interest to this algorithm. Also, $u(x) = \frac{\Gamma(a+x+1-c)}{\Gamma(a+x)} {}_2F_1(a+x+1-c, b+1-c; 2-c; z)$ is another solution of $L2F1$ when $u(x)$ is defined and $c \notin \mathbb{Z}$ by [17, Theorem 4.4].

5.3 Algorithm

Suppose we have an operator L that has $\text{Val}(L) = \{[p_1, 1], [p_2, 1]\}$ and $\text{Gquo}(L) = \{1 - d_1T + d_2T^2, 1 + d_1T + d_2T^2\}$. Then local data of L matches with those of Lkm in the table. We need to compute candidate values for the parameters of Lkm , a, b and c . d_1 will give exact values of $2\sqrt{c}T$ and d_2 will give $2c \bmod \frac{1}{2}\mathbb{Z}$. So we will use d_1 to compute

²This sequence is not a special function. However this shows that any parameterized equation with known solutions can be added to the table.

possible values for c , $c = \frac{d^2}{4}$. $(p_1, p_2) = (-a, -a + b)$ or $(p_1, p_2) = (-a + b, -a) \pmod{\mathbb{Z}}$. We need $a, b \pmod{\mathbb{Z}}$. Thus the set of candidate values for b are $\{p_1 - p_2, p_2 - p_1\}$. For the case $b = p_1 - p_2$ the set of candidate values for a are $\{-p_2, p_1 - 2p_2\}$ and for the case $b = p_2 - p_1$ the set of candidate values for a are $\{-p_1, -2p_1 + p_2\}$. In all, we will have 4 equations to check GT-equivalence. All candidates in Step 4a of the following algorithm were generated similarly.

Algorithm 5.3.1. *solver*

Input: An operator $L = a_2\tau^2 + a_1\tau + a_0 \in \mathbb{Q}(x)[\tau]$.

Output: At least one solution of L if there is an operator in the table in Section 5.2 to which L can be reduced. Otherwise the empty set.

1. By multiplying suitable polynomial in $\mathbb{Q}[x]$ let $L \in \mathbb{Q}[x][\tau]$.
2. If $\text{Tausqsols}(L) \neq \emptyset$ then return $\text{Tausqsols}(L)$.
3. Compute $\text{Gquo}(L)$ and $\text{Val}(L)$.
4. Compare $\text{Gquo}(L)$ and $\text{Val}(L)$ with those in the table and find a base equation that matches the data. If there is no such base equation then return \emptyset .
 - (a) Compute candidate values for each parameters.
 - (b) Construct a set cdd by plugging values found in step 4a to corresponding parameters.
5. For each $L_c \in cdd$ check if L can be reduced to L_c and if so
 - (a) Generate a basis of solutions or a solution of L_c by plugging in corresponding parameters.
 - (b) Apply the term transformation and the gauge transformation to the result from 5a.
 - (c) Return the result of step 5b as output and stop the algorithm.

In implemented algorithm there are more base equations which are special case of each base equations in the table given in section 5.2. Suppose $c \in \mathbb{Z}$ in $L2F1$. Since we need $c \pmod{\mathbb{Z}}$ [17, Lemma 4.3], we may let $c = 0$. Let $L2F1_1 := L2F1|_{c=0} = (z-1)(a+x+1)\tau^2 + (-z+2-za-zx+2a+2x+zb)\tau - a-1-x$. Then $\text{Val}(L2F1) = \{-a, 2\}$ and $\text{Gquo}(L2F1) = \{-\frac{1}{z-1}(1+(2b)T), (-z+1)(1+(-2b)T)\}$.

5.4 Effectiveness

To check the effectiveness of *solver*, we have found 10,659 sequences in The Online Encyclopedia of Integers Sequences [1] that satisfy a second order recurrence but not a first order recurrence. With *solver* we find

- 9,455 were reducible, (that is, there is at least one hypergeometric solution)
- 161 irreducible Liouvillian,

- 86 Bessel,
- 330 Legendre,
- 374 Hermite,
- 21 Jacobi,
- 8 Kummer,
- 44 Laguerre,
- $7 {}_2F_1$,
- $14 {}_2F_0$,
- 77 Generating function $(1+x)^a(1+bx)^c$, and
- 82 Not yet solved.

The explanation why *solver* solves so many equation is that it can detect a match up to GT-equivalence, and GT-transformations are the main order-preserving transformations. Many of the remaining 84 not yet solved equation can be treated quite easily by adding just a few more base equations to the table used by *solver*.

You can add a recurrence relation to *solver* as follows: First, compute $\text{Val}(L)$ and $\text{Gquo}(L)$ by the commands *gpmaxmin(L)* and *gquo(genexp(L))*, then add it to the table of *solver*. The implementation can be downloaded from www.math.fsu.edu/~ycha.

BIBLIOGRAPHY

- [1] The on-line encyclopedia of integer sequences. published electronically at <http://oeis.org>, 2010.
- [2] S. A. Abramov, M. A. Barkatou, and D. E. Khmelnov. On m-interlacing solutions of linear difference equations. In *CASC '09: Proceedings of the 11th International Workshop on Computer Algebra in Scientific Computing*, pages 1–17, Berlin, Heidelberg, 2009. Springer-Verlag.
- [3] S. A. Abramov, M. A. Barkatou, and M. van Hoeij. Apparent singularities of linear difference equations with polynomial coefficients. *Appl. Algebra Eng., Commun. Comput.*, 17(2):117–133, 2006.
- [4] Sergei A. Abramov and Manuel Bronstein. On solutions of linear functional systems. In *ISSAC '01: Proceedings of the 2001 international symposium on Symbolic and algebraic computation*, pages 1–6, New York, NY, USA, 2001. ACM.
- [5] Milton Abramowitz and Irene A. Stegun. *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Dover, New York, ninth dover printing, tenth gpo printing edition, 1964.
- [6] M. A. Barkatou. Computing the exponential part of a formal fundamental matrix solution of a linear difference system. *Journal of Difference Equations and Applications*, 5:117 – 142, 1999.
- [7] M. A. Barkatou. Rational solutions of matrix difference equations: the problem of equivalence and factorization. In *ISSAC '99: Proceedings of the 1999 international symposium on Symbolic and algebraic computation*, pages 277–282, New York, NY, USA, 1999. ACM.
- [8] M. A. Barkatou and G. Chen. Some formal invariants of linear difference systems and their computations. *J. Reine Angew. Math.*, 533:1–23, 2001.
- [9] G. D. Birkhoff. Formal theory of irregular linear difference equations. *Acta Math.*, 54:205–246, 1930.
- [10] Raphael Bomboy. *Réductibilité et résolubilité des équations aux différences finies*. PhD thesis, l’université de Nice, 2001.
- [11] Manuel Bronstein and Marko Petkovšek. On ore rings, linear operators and factorization. *Program. Comput. Software*, 20, 1994.

- [12] Manuel Bronstein and Marko Petkovšek. An introduction to pseudo-linear algebra. *Theor. Comput. Sci.*, 157(1):3–33, 1996.
- [13] Yongjae Cha and Mark van Hoeij. Liouvillian solutions of irreducible linear difference equations. In *ISSAC '09: Proceedings of the 2009 international symposium on Symbolic and algebraic computation*, pages 87–94, New York, NY, USA, 2009. ACM.
- [14] Yongjae Cha, Mark van Hoeij, and Giles Levy. Solving recurrence relations using local invariants. In *ISSAC '10: Proceedings of the 2010 International Symposium on Symbolic and Algebraic Computation*, pages 303–309, New York, NY, USA, 2010. ACM.
- [15] Thomas Cluzeau and Mark van Hoeij. Computing hypergeometric solutions of linear recurrence equations. *Appl. Algebra Eng., Commun. Comput.*, 17(2):83–115, 2006.
- [16] Ruyong Feng, Michael F. Singer, and Min Wu. Liouvillian solutions of linear difference-differential equations. *J. Symb. Comput.*, 45(3):287–305, 2010.
- [17] Levy Giles. *Solutions of second order recurrence relations*. PhD thesis, Florida State University, 2010.
- [18] Peter A. Hendricks and Michael F. Singer. Solving difference equations in finite terms. *J. Symb. Comput.*, 27(3):239–259, 1999.
- [19] Manuel Kauers. *Algorithms for Nonlinear Higher Order Difference Equations*. PhD thesis, RISC-Linz, Johannes Kepler University, 2005.
- [20] D. E. Khmelnov. Search for liouvillian solutions of linear recurrence equations in the maple computer algebra system. *Program. Comput. Softw.*, 34(4):204–209, 2008.
- [21] A. H. M. Levelt and A. Fahim. Characteristic classes for difference operators. *Compositio Mathematica*, 117(02):223–241, 1999.
- [22] Doron Zeilberger Marko Petkovsek, Herbert S. Wilf. *A=B*. AK Peters, Ltd., 1996.
- [23] Oystein Ore. Theory of non-commutative polynomials. *The Annals of Mathematics*, 34(3):pp. 480–508, 1933.
- [24] M. van der Put and M. F. Singer. *Galois Theory of Difference Equations*, volume 1666. Springer-Verlag, 1997.
- [25] Mark van Hoeij. Finite singularities and hypergeometric solutions of linear recurrence equations. *J. Pure Appl. Algebra*, 139:109–131, 1998.
- [26] Robert John Walker. *Algebraic curves*. Princeton, Princeton University Press, 1950.

BIOGRAPHICAL SKETCH

I was born in Seoul, Korea on January 15th, 1978 and spent my childhood in Korea, the United States and Hong Kong. I served my mandatory military service as KATUSA (Korean Augmented Troops to U.S. Army). I have started my Ph.D at Fall, 2005 and by the time I got married, I passed the qualifier. My twin sons was born around the time when I wrote my first paper.