Closed Form Solutions of Linear Difference Equations

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Object of the Thesis:
Algorithm *solver* that solves difference operators.

1. Transformations
2. Invariant Data
3. Table of base equations

Outline

Linear Difference Equation

- Difference Equation:
  Let $DE : \mathbb{C}^{n+2} \to \mathbb{C}$. Then a difference equation is an equation of the form

$$DE(f(x), f(x+1), \ldots, f(x+n), x) = 0 \quad (n \geq 1)$$

- A recurrence relation
  Let $R : \mathbb{C}^{n+1} \to \mathbb{C}$. Then a recurrence relation is an equation of the form

$$f(x+n) = R(f(x), f(x+1), \ldots, f(x+n-1), x) \quad (n \geq 1)$$

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Linear Difference Equation

A difference equation is called linear, if it is in the form of

$$a_n(x)f(x+n) + a_{n-1}(x)f(x+n-1) + \cdots + a_0(x)f(x) + a(x) = 0$$

where $a, a_i : \mathbb{C} \to \mathbb{C}$ for $i = 0, \ldots, n$.
Then it naturally defines a recurrence relation by

$$f(x+n) = -\frac{a_{n-1}(x)}{a_n(x)}f(x+n-1) - \cdots - \frac{a_0(x)}{a_n(x)}f(x) - \frac{a(x)}{a_n(x)}$$

A difference equation is called homogeneous if $a(x) = 0$.

In this talk we will only consider homogeneous linear difference equations with coefficients in $\mathbb{C}(x)$.

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Linear Difference Operator

Let τ be the shift operator: $\tau(u(x)) = u(x+1)$
Then a Linear Difference Operator L is

$$L = a_n\tau^n + a_{n-1}\tau^{n-1} + \cdots + a_0\tau^0 \text{ where } a_i \in \mathbb{C}(x).$$

L corresponds to a difference equation

$$a_n(x)f(x+n) + a_{n-1}(x)f(x+n-1) + \cdots + a_0(x)f(x) = 0.$$

Example:

- If $L = \tau - x$ then the equation $L(f(x)) = 0$ is $f(x+1) - xf(x) = 0$ and $\Gamma(x)$ is a solution of L.

We will see some examples of what *solver* can do.
(with Maple worksheet)

GT-Transformation

Notation:

- $V(L)$ = solution space of L.

1. Term Product: L_2 is a term product of L_1 when $V(L_2)$ can be written as $V(L_1)$ multiplied by a hypergeometric term.
2. Gauge Equivalence: L_2 is gauge equivalent to L_1 if there exists $G \in \mathbb{C}(x)[\tau]$ that bijectively maps $V(L_1)$ to $V(L_2)$.
3. GT-Equivalence: $L_2 \sim_{gt} L_1$ if a combination of (1) and (2) can map $V(L_1)$ to $V(L_2)$. Such map is called GT-Transformation.

We can find GT-Transformation.

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Main Idea

Observation:
If two operators are gt-equivalent and if one of them has closed form solutions, then so does the other.

Idea:

- Find base equations: Find parameterized families of equations with known solutions.
- Solve every equation $\sim_{gt}$ to a base equation.

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1. Can we construct such table?

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- $LbIK = z\tau^2 + (2 + 2v + 2x)\tau - z$
  Solutions: Modified Bessel functions of the first and second kind, $I_{v+x}(z)$ and $K_{v+x}(-z)$
- $LbJY = z\tau^2 - (2 + 2v + 2x)\tau + z$
  Solutions: Bessel functions of the first and second kind, $J_{v+x}(z)$ and $Y_{v+x}(z)$
- $LWW = \tau^2 + (z - 2v - 2x - 2)\tau - v - x - \frac{1}{4} - v^2 - 2vx - x^2 + n^2$
  Solution: Whittaker function $W_{x,n}(z)$
- $LWM = \tau^2(2n + 2v + 3 + 2x) + (2z - 4v - 4x - 4)\tau - 2n + 1 + 2v + 2x$
  Solution: Whittaker function $M_{x,n}(z)$
- $L2F1 = (z - 1)(a + x + 1)\tau^2 + (-z + 2 - za - zx + 2a + 2x + zb - c)\tau - a + c - 1 - x$
  Solution: Hypergeometric function ${}_2F_1(a + x, b; c; z)$
- $Ljc = \tau^2 - \frac{1}{2}\frac{(2x+3+a+b)(a^2-b^2+(2x+a+b+2)(2x+4+a+b)z)}{(x+2)(x+2+a+b)(2x+a+b+2)}\tau + \frac{(x+1+a)(x+1+b)(2x+4+a+b)}{(x+2)(x+2+a+b)(2x+a+b+2)}$
  Solution: Jacobian polynomial $P_x^{a,b}(z)$
- $Lgd = \tau^2 - \frac{(2x+3)z}{x+2}\tau + \frac{x+1}{x+2}$
  Solution: Legendre functions $P_x(z)$ and $Q_x(z)$
- $Lgr = \tau^2 - \frac{2x+3+\alpha-z}{x+2}\tau + \frac{x+1+\alpha}{x+2}$
  Solution: Laguerre polynomial $L_x^{(\alpha)}(z)$
- $Lgb = \tau^2 - \frac{2z(m+x+1)}{x+2}\tau - \frac{2m+x}{x+2}$
  Solution: Gegenbauer polynomial $C_x^m(z)$
- $Lgr1 = (x + 2)\tau^2 + (x + z - b + 1)\tau + z$
  Solution: Laguerre polynomial $L_x^{(b-x)}(z)$
- $Lkm = (a + x + 1)\tau^2 + (-2a - 2x - 2 + b - c)\tau + a + x + 1 - b$
  Solution: Kummer's function $M(a + x, b, c)$
- $L2F0 = \tau^2 + (-zb + zx + z + za - 1)\tau + z(b - x - 1)$
  Solution: Hypergeometric function ${}_2F_0(a, b - x; ; z)$
- $Lge = (x + 2)\tau^2 + (-ab - d + (a + 1)(1 + x))\tau + ax - a(b + d)$
  Solution: Sequences whose ordinary generating function is $(1 + ax)^b(1 + bx)^d$

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2. How can we find the right base equation and the parameter values?

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Main Algorithm

1. Compute local data of L.
2. Compare the data with those in the table and find a base equation that matches the data. If there is no such base equation then return $\emptyset$.
   1. Compute candidate values for each parameters.
   2. Construct a set *cdd* by plugging values found in step 1 to corresponding parameters.
3. For each $L_c \in cdd$ check if $L \sim_{gt} L_c$ and if so
   1. Generate a basis of solutions or a solution of L_c by plugging in corresponding parameters.
   2. Apply the term transformation and the gauge transformation to the result from 1.
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Algorithm

TryBessel:
Input: $L \in \mathbb{C}(x)[\tau]$

1. Compute the local data of $L_{v,z} = z\tau^2 + (2 + 2v + 2x)\tau - z$ (Bessel recurrence).
2. Compute local data of L that is invariant under $\sim_{gt}$.
3. Compare the local data of $L_{v,z}$ with that of L.
4. If compatible, compute v, z from this comparison.
5. Check if $L \sim_{gt} L_{v,z}$, and if so, return solution(s).

Note: Step 1 is done only once, and then stored in a table.

Remark: Checking $L \sim_{gt} L_{v,z}$ and computing the gt-transformation can only be done after we have found the values of the parameter v, z.

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Invariant Local Data

Question: If $L \sim_{gt} L_{v,z}$, how to find v, z from L?

Need data that is invariant under $\sim_{gt}$

Two sources

1. Finite Singularities (valuation growths)
2. Singularity at ∞ (generalized exponents)

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Finite Singularity: Valuation Growth

Suppose $L_1 \sim_g L_2$ and $G = r_k(x)\tau^k + \cdots + r_0(x), r_i(x) \in \mathbb{C}(x)$
Let $u(x) = \Gamma(x) \in V(L_1)$ and $v(x) = G(u(x))$ is a non-zero element in $V(L_2)$.

$\mathbb{Z}$	-3	-2	-1	0	1	2	3
valuation of $u(x)$	-1	-1	-1	-1	0	0	0

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$\mathbb{Z}$	p_l-3	p_l-2	p_l-1	- - - -	p_r+1	p_r+2	p_r+3
valuation of $v(x)$	-1	-1	-1		0	0	0

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$$a_n(x)f(x+n)+a_{n-1}(x)f(x+n-1)+\cdots+a_0(x)f(x)+a(x)=0 \quad a_i(x)\in\mathbb{C}[x].$$

To calculate $f(s+n)$ with values of $f(s),\ldots,f(s+n-1)$, $s\in\mathbb{C}$,

$$f(x+n)=-\frac{a_{n-1}(x)}{a_n(x)}f(x+n-1)-\cdots-\frac{a_0(x)}{a_n(x)}f(x)$$

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$$f(x)=-\frac{a_n(x)}{a_0(x)}f(x+n)-\cdots-\frac{a_1(x)}{a_0(x)}f(x+1)$$

Definition

Let $L=a_n\tau^n+\cdots+a_0\tau^0$ with $a_i\in\mathbb{C}[x]$. $q\in\mathbb{C}$ is called a *problem point* of L if q is a root of the polynomial $a_0(x)a_n(x-n)$. $p\in\mathbb{C}/\mathbb{Z}$ is called a *finite singularity* of L if it contains a problem point.

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Definition

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$$\liminf_{n\to\infty}(\text{order of } u(x) \text{ at } x = n+q) - \liminf_{n\to\infty}(\text{order of } u(x) \text{ at } x = -n+q)$$

Definition

Let $p \in \mathbb{C}/\mathbb{Z}$ and L be a difference operator. Then $\mathrm{Min}_p(L)$ resp. $\mathrm{Max}_p(L)$ is the minimum resp. maximum valuation growth at p, taken over all meromorphic solutions of L.

Theorem

If $L_1 \sim_g L_2$ then they have the same $\mathrm{Min}_p, \mathrm{Max}_p$ for all $p \in \mathbb{C}/\mathbb{Z}$.

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$\mathrm{Max}_p - \mathrm{Min}_p$ is $\sim_{gt}$ invariant for all $p \in \mathbb{C}/\mathbb{Z}$.

Invariant data: Compute all $p \in \mathbb{C}/\mathbb{Z}$ for which $\mathrm{Max}_p \neq \mathrm{Min}_p$ store $[p, \mathrm{Max}_p - \mathrm{Min}_p]$ for all such p.

Note: Since $p \in \mathbb{C}/\mathbb{Z}$ and not in $\mathbb{C}$, the parameters computed from such data are determined mod $r\mathbb{Z}$ for some $r \in \mathbb{Q}$. Suppose we need parameter ν mod $\mathbb{Z}$ but find it mod $\frac{1}{2}\mathbb{Z}$, then we need to check two cases.

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Singularity at ∞ : Generalized Exponent

Definition

If $\tau - ct^v(1 + \sum_{i=1}^{\infty} a_i t^{\frac{i}{r}})$, with $t = 1/x$, is right hand factor of L for some $v \in \frac{1}{r}\mathbb{Z}, c \in \mathbb{C}^*, a_i \in \mathbb{C}, r \in \mathbb{N}$, then the dominant term $ct^v(1 + a_1 t^{\frac{1}{r}} + \cdots + a_r t^1)$ is called a *generalized exponent* of L.

We say two generalized exponents $g_1 = c_1 t^{v_1}(1 + a_1 t^{\frac{1}{r}} + \cdots + a_r t^1)$ and $g_2 = c_2 t^{v_2}(1 + b_1 t^{\frac{1}{r}} + \cdots + b_r t^1)$ are equivalent if $c_1 = c_2, v_1 = v_2, a_i = b_i$ for $i = 1 \ldots r - 1$ and $a_r \equiv b_r \mod \frac{1}{r}\mathbb{Z}$ and denote $g_1 \sim_r g_2$

Theorem

Generalized exponents are invariant up to $\sim_r$ under Gauge equivalence.

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Singularity at ∞ : Generalized Exponent

Generalized exponents are not invariant under term-product.

Definition

Suppose $ord(L) = 2$ and let $\mathrm{genexp}(L) = \{a_1, a_2\}$ such that $v(a_1) \geq v(a_2)$. Then we define the *set of quotient of the two generalized exponents* as
if $v(a_1) > v(a_2)$

$$\mathrm{Gquo}(L) = \left\{\frac{a_1}{a_2}\right\} \text{ and}$$

if $v(a_1) = v(a_2)$ then we define

$$\mathrm{Gquo}(L) = \left\{\frac{a_1}{a_2}, \frac{a_2}{a_1}\right\}.$$

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If $L_1 \sim_{gt} L_2$ *then* $\mathrm{Gquo}(L_1) = \mathrm{Gquo}(L_2) \bmod \sim_r$

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Outline

Liouvillian Solutions of Linear Difference Equations: Property

Theorem (Hendriks Singer 1999)

If $L = a_n\tau^n + \cdots + a_0\tau^0$ *is irreducible then*

$\exists$ *Liouvillian Solutions* $\Longleftrightarrow \exists b_0 \in \mathbb{C}(x)$ *such that*

$$a_n\tau^n + \cdots + a_0\tau^0 \quad \sim_g \quad \tau^n + b_0\tau^0$$

Remark

Operators of the form $\tau^n + b_0\tau^0$ are easy to solve, so if we know b_0 then we can solve L.

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Let $L = a_n\tau^n + \cdots + a_0\tau^0$ with $a_i \in \mathbb{C}[x]$ and assume that

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for some unknown $b_0 \in \mathbb{C}(x)$.

If we can find b_0 then we can solve $\tau^n + b_0\tau^0$ and hence solve L.

Notation

write $b_0 = c\phi$ where $\phi = \frac{\text{monic poly}}{\text{monic poly}}$ and $c \in \mathbb{C}^*$.

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c is easy to compute, the main task is to compute ϕ.

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Let $L = a_n\tau^n + \cdots + a_0\tau^0 \in \mathbb{C}[x][\tau]$ then the finite singularities of L are $Sing = \{q + \mathbb{Z} \in \mathbb{C}/\mathbb{Z} \mid \text{q is root of } a_0 a_n\}$

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If $q_1 + \mathbb{Z}, \ldots, q_k + \mathbb{Z}$ are the finite singularities then we may assume $\phi = \prod_{i=1}^{k}\prod_{j=0}^{n-1}(x - q_i - j)^{k_{i,j}}$ *with* $k_{i,j} \in \mathbb{Z}$.

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Example of Operator of order 2 with one finite singularity at $p = \mathbb{Z}$

Suppose $L = a_2\tau^2 + a_1\tau + a_0$ and that

$$L \sim_g \tau^2 + c \cdot x^{k_0}(x-1)^{k_1}$$

1. c can be computed from a_0/a_2
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Items 2, 3, 4 determine k_0, k_1 up to a permutation.

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- $\tau^2 - \tau - x(x+1)$
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Example with two finite singularities at $\mathbb{Z}$ and $\frac{1}{2}+\mathbb{Z}$

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- At $\frac{1}{2} + \mathbb{Z}$,

$$min = 0, \quad max = 1, \quad sum = 1$$

So the exponents of $(x - \frac{1}{2})^{\cdots}(x - \frac{3}{2})^{\cdots}(x - \frac{5}{2})^{\cdots}$ must be a permutation of $0, 0, 1$

Liouvillian Solutions of Linear Difference Equations: Example $L = x\tau^3 + \tau^2 - (x+1)\tau - x(x+1)^2(2x-1)$

Candidates of $c\phi$ are

1. $-2x^1(x-1)^1(x-2)^0(x-1/2)^1(x-3/2)^0(x-5/2)^0$
2. $-2x^1(x-1)^1(x-2)^0(x-1/2)^0(x-3/2)^1(x-5/2)^0$
3. $-2x^1(x-1)^1(x-2)^0(x-1/2)^0(x-3/2)^0(x-5/2)^1$
4. $-2x^0(x-1)^1(x-2)^1(x-1/2)^0(x-3/2)^0(x-5/2)^1$
5. $-2x^0(x-1)^1(x-2)^1(x-1/2)^0(x-3/2)^1(x-5/2)^0$
6. $-2x^0(x-1)^1(x-2)^1(x-1/2)^1(x-3/2)^0(x-5/2)^0$
7. $-2x^1(x-1)^0(x-2)^1(x-1/2)^1(x-3/2)^0(x-5/2)^0$
8. $-2x^1(x-1)^0(x-2)^1(x-1/2)^0(x-3/2)^0(x-5/2)^1$
9. $-2x^1(x-1)^0(x-2)^1(x-1/2)^0(x-3/2)^1(x-5/2)^0$

Liouvillian Solutions of Linear Difference Equations: Example $L = x\tau^3 + \tau^2 - (x+1)\tau - x(x+1)^2(2x-1)$

Remark

$$\tau^n - c\phi \sim_g \tau^n - c\tau^k(\phi) \text{ for } k = 1 \ldots n-1$$

1. $-2x^1(x-1)^1(x-2)^0(x-1/2)^1(x-3/2)^0(x-5/2)^0$
2. $-2x^1(x-1)^1(x-2)^0(x-1/2)^0(x-3/2)^1(x-5/2)^0$
3. $-2x^1(x-1)^1(x-2)^0(x-1/2)^0(x-3/2)^0(x-5/2)^1$

Only need to try 1, 2, 3, the others are redundant.

Liouvillian Solutions of Linear Difference Equations: Example $L = x\tau^3 + \tau^2 - (x+1)\tau - x(x+1)^2(2x-1)$

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Only need to try 1, 2, 3, the others are redundant.

Liouvillian Solutions of Linear Difference Equations: Example $L = x\tau^3 + \tau^2 - (x+1)\tau - x(x+1)^2(2x-1)$

- $\tau^3 - 2x(x-1)(x-1/2)$ is gauge equivalent to L
- Gauge transformation is $\tau + x - 1$.
- Basis of solutions of $\tau^3 - 2x(x-1)(x-1/2)$ is

$$\{(\xi^k)^x v(x)\} \quad \text{for} \quad k = 0 \ldots 2$$

where $v(x) = 3^x 2^{x/3}\Gamma(\frac{x}{3})\Gamma(\frac{x-1}{3})\Gamma(\frac{x-\frac{1}{2}}{3})$ and $\xi^3 = 1$.

- Thus, Basis of solutions of L is

$$\{(\xi^k)^{x+1} v(x+1) + (x-1)(\xi^k)^x v(x)\} \quad \text{for} \quad k = 0 \ldots 2$$

Liouvillian Solutions of Linear Difference Equations: Example $L = x\tau^3 + \tau^2 - (x+1)\tau - x(x+1)^2(2x-1)$

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- Thus, Basis of solutions of L is

$$\{(\xi^k)^{x+1} v(x+1) + (x-1)(\xi^k)^x v(x)\} \quad \text{for} \quad k = 0 \ldots 2$$

Outline

- $LbIK = z\tau^2 + (2+2v+2x)\tau - z$
  Solutions: Modified Bessel functions of the first and second kind, $I_{v+x}(z)$ and $K_{v+x}(-z)$
- $LbJY = z\tau^2 - (2+2v+2x)\tau + z$
  Solutions: Bessel functions of the first and second kind, $J_{v+x}(z)$ and $Y_{v+x}(z)$
- $LWW = \tau^2 + (z-2v-2x-2)\tau - v - x - \frac{1}{4} - v^2 - 2vx - x^2 + n^2$
  Solution: Whittaker function $W_{x,n}(z)$
- $LWM = \tau^2(2n+2v+3+2x) + (2z-4v-4x-4)\tau - 2n + 1 + 2v + 2x$
  Solution: Whittaker function $M_{x,n}(z)$
- $L2F1 = (z-1)(a+x+1)\tau^2 + (-z+2-za-zx+2a+2x+zb-c)\tau - a + c - 1 - x$
  Solution: Hypergeometric function ${}_2F_1(a+x, b; c; z)$
- $Ljc = \tau^2 - \frac{1}{2}\frac{(2x+3+a+b)(a^2-b^2+(2x+a+b+2)(2x+4+a+b)z)}{(x+2)(x+2+a+b)(2x+a+b+2)}\tau + \frac{(x+1+a)(x+1+b)(2x+4+a+b)}{(x+2)(x+2+a+b)(2x+a+b+2)}$
  Solution: Jacobian polynomial $P_x^{a,b}(z)$
- $Lgd = \tau^2 - \frac{(2x+3)z}{x+2}\tau + \frac{x+1}{x+2}$
  Solution: Legendre functions $P_x(z)$ and $Q_x(z)$
- $Lgr = \tau^2 - \frac{2x+3+\alpha-z}{x+2}\tau + \frac{x+1+\alpha}{x+2}$
  Solution: Laguerre polynomial $L_x^{(\alpha)}(z)$
- $Lgb = \tau^2 - \frac{2z(m+x+1)}{x+2}\tau - \frac{2m+x}{x+2}$
  Solution: Gegenbauer polynomial $C_x^m(z)$
- $Lgr1 = (x+2)\tau^2 + (x+z-b+1)\tau + z$
  Solution: Laguerre polynomial $L_x^{(b-x)}(z)$
- $Lkm = (a+x+1)\tau^2 + (-2a-2x-2+b-c)\tau + a + x + 1 - b$
  Solution: Kummer's function $M(a+x, b, c)$
- $L2F0 = \tau^2 + (-zb+zx+z+za-1)\tau + z(b-x-1)$
  Solution: Hypergeometric function ${}_2F_0(a, b-x; ; z)$
- $Lge = (x+2)\tau^2 + (-ab-d+(a+1)(1+x))\tau + ax - a(b+d)$
  Solution: Sequences whose ordinary generating function is $(1+ax)^b(1+x)^d$

Special Functions: Functions and their Local Data

Operator	Val	Gquo
$LbIK$	$\{\}$	$\{-\frac{1}{4}T^2z^2(1-(1+2v)T)\}$
$LbJY$	$\{\}$	$\{\frac{1}{4}T^2z^2(1-(1+2v)T)\}$
LWW	$\{[-n+\frac{1}{2}-v,1],[n+\frac{1}{2}-v,1]\}$	$\{-3-2\sqrt{2}(1-\frac{1}{2}\sqrt{2}z)T,-3+2\sqrt{2}(1+\frac{1}{2}\sqrt{2}z)T\}$
LWM	$\{[-n+\frac{1}{2}-v,1],[n+\frac{1}{2}-v,1]\}$	$\{1-2\sqrt{-z}T-2zT^2,1+2\sqrt{-z}T-2zT^2\}$
$L2F1$	$\{[-a+c,1],[-a,1]\}$	$\{-\frac{1}{z-1}(1+(2b-c)T),(-z+1)(1+(-2b+c)T)\}$
Ljc	$\{[0,1],[-a,1],[-b,1],$ $[-a-b,1]\}$	$\{2z^2-2z\sqrt{z^2-1}-1,2z^2+2z\sqrt{z^2-1}-1\}$
Lgd	$\{[0,2]\}$	$\{2z^2-2z\sqrt{z^2-1}-1,2z^2+2z\sqrt{z^2-1}-1\}$
Lgr	$\{[0,1],[-\alpha,1]\}$	$\{1+2\sqrt{-z}T-2zT^2,1-2\sqrt{-z}T-2zT^2\}$
$Lgr1$	$\{[0,1]\}$	$\{zT(1+2bT)\}$
Lgb	$\{[0,1],[-2m,1]\}$	$\{-2z\sqrt{z^2+1}-2z^2-1,2z\sqrt{z^2+1}-2z^2-1\}$
Lkm	$\{[-a,1],[-a+b,1]\}$	$\{1-2\sqrt{c}T+2cT^2,1+2\sqrt{c}T+2cT^2\}$
$L2F0$	$\{[b,1]\}$	$\{\frac{T}{z}(1+(b-2a)T)\}$
Lge	$\{[0,1],[b+d,1]\}$	$\{a(1+(d-b)T),\frac{1}{a}(1+(-b-d)T)\}$

Effectiveness of *solver*

Found 10,659 sequences in OEIS™ that satisfy a second order recurrence but not a first order recurrence.

- 9,455 were reducible
- 161 irreducible Liouvillian
- 86 Bessel
- 330 Legendre
- 374 Hermite
- 21 Jacobi
- 8 Kummer
- 44 Laguerre
- 7 ${}_2F_1$
- 14 ${}_2F_0$
- 77 Generating function $(1+x)^a(1+bx)^c$
- 82 Not yet solved

Example from

The On-Line Encyclopedia of Integer Sequences™(OEIS™)

A096121 = (2, 8, 60, 816, 17520, 550080, 23839200,...)
Number of full spectrum rook's walks on a (2 x n) board.

- Difference Operator: $\tau^2 - (1+x)(x+2)\tau - (1+x)(x+2)$
- Val: $\{\}$
- Gquo: $\{-T^2(1-3T)\}$

Modified Bessel functions of the first and second kind, $I_{v+x}(z)$ and $K_{v+x}(-z)$.

- Difference Operator: $z\tau^2 + (2+2v+2x)\tau - z$
- Val: $\{\}$
- Gquo: $\{-\frac{1}{4}T^2z^2(1-(1+2v)T)\}$

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- Val: $\{\}$
- Gquo: $\{-\frac{1}{4}T^2z^2(1 - (1 + 2v)T)\}$

Example from

The On-Line Encyclopedia of Integer Sequences™ (OEIS™)

- Comparing Gquo,
  $\{-T^2(1-3T)\}$ and $\{-\frac{1}{4}z^2T^2(1-(1+2v)T)\}$,
  we get candidates of $z = \{2, -2\}$
  and candidates of $v = \{\frac{1}{2}, 1\}$
- We get four candidates to check $\sim_{gt}$,

$$2\tau^2 - (2x+4)\tau - 2,\ 2\tau^2 - (2x+3)\tau - 2$$

$$2\tau^2 + (2x+4)\tau - 2,\ 2\tau^2 + (2x+3)\tau - 2.$$

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Example from

The On-Line Encyclopedia of Integer Sequences™(OEIS™)

- $2\tau^2 - (2x+4)\tau - 2 \sim_{gt} L$ (When $v = 1, z = -2$)
- Term-transformation is $x + 2$ and gauge-transformation is 1.
- Applying gt-transformation to $I_{1+x}(2)$ and $K_{1+x}(-2)$ we get basis of a basis of solutions of L,

$$\{I_{1+x}(2)\Gamma(x+2), K_{1+x}(-2)\Gamma(x)\}$$

Example from

The On-Line Encyclopedia of Integer Sequences[TM](OEIS[TM])

- $2\tau^2 - (2x+4)\tau - 2 \sim_{gt} L$ (When $v = 1, z = -2$)
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$$\{I_{1+x}(2)\Gamma(x+2), K_{1+x}(-2)\Gamma(x)\}$$

Example from

The On-Line Encyclopedia of Integer Sequences™(OEIS™)

A000262 = (1, 1, 3, 13, 73, 501, 4051, 37633, 394353,...)
Number of "sets of lists":
number of partitions of $\{1, .., n\}$ into any number of lists.

- Difference Operator: $\tau^2 - (3+2x)\tau + x(x+1)$
- Val: $\{[0,2]\}$
- Gquo: $\{1 - 2T + 2T^2, 1 + 2T + 2T^2\}$

Laguerre polynomial $L_x^{(\alpha)}(z)$.

- Difference Operator:$Lgr = \tau^2 - \frac{2x+3+\alpha-z}{x+2}\tau + \frac{x+1+\alpha}{x+2}$
- Val: $\{[0,1],[-\alpha,1]\}$
- Gquo: $\{1 - 2\sqrt{-z}T - 2zT^2, 1 + 2\sqrt{-z}T - 2zT^2)\}$

Example from

The On-Line Encyclopedia of Integer Sequences™(OEIS™)

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Laguerre polynomial $L_x^{(\alpha)}(z)$.

- Difference Operator:$Lgr = \tau^2 - \frac{2x+3+\alpha-z}{x+2}\tau + \frac{x+1+\alpha}{x+2}$
- Val: $\{[0, 1], [-\alpha, 1]\}$
- Gquo: $\{1 - 2\sqrt{-z}T - 2zT^2, 1 + 2\sqrt{-z}T - 2zT^2)\}$

Example from

The On-Line Encyclopedia of Integer Sequences™(OEIS™)

- Comparing Gquo,
  $\{1 - 2T + 2T^2 \quad 1 + 2T + 2T^2\}$ and
  $\{1 - 2\sqrt{-z}T - 2zT^2, 1 + 2\sqrt{-z}T - 2zT^2)\}$,
  we get $z = -1$.
- Val$= \{[0, 2]\}$ is a special case of *Lgr* when $\alpha = 0$.
- We get one candidate to check $\sim_{gt}$,

$$\tau^2 - \frac{2x+4}{x+2}\tau + \frac{x+1}{x+2}$$

Example from

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  $\{1 - 2T + 2T^2 \quad 1 + 2T + 2T^2\}$ and
  $\{1 - 2\sqrt{-z}T - 2zT^2, 1 + 2\sqrt{-z}T - 2zT^2)\}$,
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- Val$=\{[0,2]\}$ is a special case of *Lgr* when $\alpha=0$.
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$$\tau^2-\frac{2x+4}{x+2}\tau+\frac{x+1}{x+2}$$

Example from

The On-Line Encyclopedia of Integer Sequences™(OEIS™)

- Term-transformation is x and gauge-transformation is $\frac{x+1}{x}\tau - \frac{x^2+2x}{x}$.
- Applying gt-transformation to $L_x^{(0)}(-1)$,

$$\{(x+1)L_{x+1}^{(0)}(-1) - (x+2)L_x^{(0)}(-1)\}\Gamma(x)$$

Example from

The On-Line Encyclopedia of Integer Sequences™(OEIS™)

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Example from
The On-Line Encyclopedia of Integer Sequences™(OEIS™)

A068770 = (1, 1, 16, 264, 4480, 77952, 1386496, 25135616,...)
Generalized Catalan numbers.

- Difference Operator: $(3+x)\tau^2 + (-48-32x)\tau + 224x$
- Val: $\{[0,2]\}$
- Gquo: $\{\frac{9}{7} - \frac{4}{7}\sqrt{2}, \frac{9}{7} + \frac{4}{7}\sqrt{2}\}$

Jacobian polynomial $P_x^{a,b}(z)$

- Difference Operator:$Lgd = \tau^2 - \frac{(2x+3)z}{x+2}\tau + \frac{x+1}{x+2}$
- Val: $\{[0,2]\}$
- Gquo: $\{2z^2 - 2z\sqrt{z^2-1} - 1, 2z^2 + 2z\sqrt{z^2-1} - 1\}$

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Example from

The On-Line Encyclopedia of Integer Sequences™ (OEIS™)

- Comparing Gquo,
  $\{\frac{9}{7} - \frac{4}{7}\sqrt{2}, \frac{9}{7} + \frac{4}{7}\sqrt{2}\}$ and
  $\{2z^2 - 2z\sqrt{z^2-1} - 1, 2z^2 + 2z\sqrt{z^2-1} - 1\}$,
  we get candidates of $z = \{\frac{2}{7}\sqrt{14}, -\frac{2}{7}\sqrt{14}\}$.
- Val$= \{[0,2]\}$ is used to find the right base equation.

We get 2 candidate to check $\sim_{gt}$,

$$\tau^2 - \frac{2}{7}\frac{(2x+3)\sqrt{14}}{x+2}\tau + \frac{1+x}{x+2}$$

$$\tau^2 + \frac{2}{7}\frac{(2x+3)\sqrt{14}}{x+2}\tau - \frac{1+x}{x+2}$$

Example from

The On-Line Encyclopedia of Integer Sequences™(OEIS™)

- Comparing Gquo, $\{\frac{9}{7}-\frac{4}{7}\sqrt{2}, \frac{9}{7}+\frac{4}{7}\sqrt{2}\}$ and $\{2z^2-2z\sqrt{z^2-1}-1, 2z^2+2z\sqrt{z^2-1}-1\}$, we get candidates of $z=\{\frac{2}{7}\sqrt{14}, -\frac{2}{7}\sqrt{14}\}$.
- Val$=\{[0,2]\}$ is used to find the right base equation.

We get 2 candidate to check $\sim_{gt}$,

$$\tau^2-\frac{2}{7}\frac{(2x+3)\sqrt{14}}{x+2}\tau+\frac{1+x}{x+2}$$

$$\tau^2+\frac{2}{7}\frac{(2x+3)\sqrt{14}}{x+2}\tau-\frac{1+x}{x+2}$$

Example from

The On-Line Encyclopedia of Integer Sequences™(OEIS™)

- $\tau^2 - \frac{2}{7}\frac{(2x+3)\sqrt{14}}{x+2}\tau + \frac{1+x}{x+2}$ (When $z = \frac{2}{7}\sqrt{14}$)
- Term-transformation is $4\sqrt{14}$ and gauge-transformation is $\frac{1}{x}(\tau - 16)$.
- Applying gt-transformation to $\{P_x(\frac{2}{7}\sqrt{14}), Q_x(\frac{2}{7}\sqrt{14})\}$, we get $\{-\frac{1}{x}(4^{x+2}14^{\frac{1}{2}x}P_x(\frac{2}{7}\sqrt{14}) + 4^{x+1}14^{\frac{1}{2}x+\frac{1}{2}}P_{x+1}(\frac{2}{7}\sqrt{14}), -\frac{1}{x}(4^{x+2}14^{\frac{1}{2}x}Q_x(\frac{2}{7}\sqrt{14}) + 4^{x+1}14^{\frac{1}{2}x+\frac{1}{2}}Q_{x+1}(\frac{2}{7}\sqrt{14})\}$

Example from

The On-Line Encyclopedia of Integer Sequences™(OEIS™)

- $\tau^2 - \frac{2}{7}\frac{(2x+3)\sqrt{14}}{x+2}\tau + \frac{1+x}{x+2}$
  (When $z = \frac{2}{7}\sqrt{14}$)
- Term-transformation is $4\sqrt{14}$ and gauge-transformation is $\frac{1}{x}(\tau - 16)$.
- Applying gt-transformation to $\{P_x(\frac{2}{7}\sqrt{14}), Q_x(\frac{2}{7}\sqrt{14})\}$, we get
  $\{-\frac{1}{x}(4^{x+2}14^{\frac{1}{2}x}P_x(\frac{2}{7}\sqrt{14}) + 4^{x+1}14^{\frac{1}{2}x+\frac{1}{2}}P_{x+1}(\frac{2}{7}\sqrt{14}),$
  $-\frac{1}{x}(4^{x+2}14^{\frac{1}{2}x}Q_x(\frac{2}{7}\sqrt{14}) + 4^{x+1}14^{\frac{1}{2}x+\frac{1}{2}}Q_{x+1}(\frac{2}{7}\sqrt{14})\}$

How to add a new base equation

One advantage of *solver* is we can add base equation to it.
(Back to Maple Worksheet)