# Solving Recurrence Relations using Local Invariants


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## ABSTRACT

The goal in this paper is to find closed form solutions for linear recurrence equations, by transforming an input equation $L$ to an equation $L_s$ with known solutions. The main problem is how to find a solved equation $L_s$ to which $L$ can be reduced. We solve this problem by computing local data at singularities, data that remains invariant under the transformations used.



## Categories and Subject Descriptors

G.2.1 [**Combinatorics**]: [Recurrences and difference equations]; I.1.2 [**Symbolic and Algebraic Manipulation**]: Algorithms—*Algebraic algorithms*

## General Terms

Algorithms


## 1. INTRODUCTION

Let $L = a_n\tau^n + \cdots + a_0\tau^0$ denote a linear difference operator. Here $\tau$ denotes the shift operator $\tau(f(x)) = f(x+1)$ and the $a_i$ are rational functions in $x$ (after multiplying away the denominators, we may assume that the $a_i$ are in $\mathbb{C}[x]$). Now $L$ corresponds to the recurrence relation $a_n u(x+n) + \cdots + a_0 u(x+0) = 0$.

This paper presents a new approach to finding solutions to linear recurrence relations with polynomial coefficients. The general approach is to transform an equation to a previously solved equation.

Given two linear recurrence operators $L_1$ and $L_2$ of the same order with coefficients from $\mathbb{C}[x]$, we give an algorithm which finds (if it exists) a map, in terms of so-called gauge transformations and term transformations, which sends the solutions of $L_1$ bijectively to $L_2$. Hence, if closed form solutions of $L_1$ are known, one obtains closed form solutions of $L_2$. Due to the fact that in the literature many special functions are defined by recurrence relations involving parameters (like orthogonal polynomials, Bessel functions of the first and second kind), we consider the following refined version: given a linear recurrence, extract information in terms of the free parameters such that any term and/or gauge transformation keeps this expression unchanged; this information is also called invariant data. In this way, one can construct a table of special functions together with its defining recurrence relations and its invariant data (w.r.t. the free parameters). One can efficiently determine for a given recurrence relation (whose solutions are unknown) the possible recurrence candidates together with the appropriate choices of the free parameters. Then for the derived choices one can check step by step if a transformation exists. This mechanism has been worked out in detail for linear recurrences of order 2.

In our previous paper [7] the table consisted of a single equation, namely $\tau^n + b$, but this equation contained a parameter $b \in \mathbb{C}(x)$, so this one equation represents an infinite set of equations, parametrized by $b \in \mathbb{C}(x)$. Now before we can compute, if it exists, a transformation between the input equation $L$ and an equation of the form $\tau^n + b$, we first have to compute the unknown $b \in \mathbb{C}(x)$. Most of [7] is devoted to finding a finite list of candidates for $b$.

In [7], the invariant data was the finite singularities. In this paper, in addition to finite singularities, we shall also use local data at infinity (generalized exponents). The reason for using both in this paper is the following: The more solved equations we add to the table, the stronger the solver will become; however, we can only add equations to the table if the parameters in those equations can be computed from the invariant data that we compute.

This means that the more invariant data we compute, the stronger we can make the solver. So besides using the implementation for finite singularities that was used in [7] (and in earlier papers [6]) we also implemented an algorithm to compute generalized exponents at the point $x = \infty$.

The mathematics of these generalized exponents has been treated in [11][1], and computationally, one can view generalized exponents as a portion of formal solutions, a topic for which algorithms have been developed [4]. Therefore, one can compute generalized exponents by implementing, as we did, a portion of an algorithm [4] to compute formal solutions. Although this mathematics is known [4, 11], it is not widely known, and so, for the convenience of the reader, we included an appendix on this topic. What is new in this paper is not the concept of generalized exponents itself, but


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[1]the characteristic classes in [11] are the minimal polynomials of what are called "generalized exponents" in this paper



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rather the way this concept is used. We use it not to compute local solutions (formal solutions containing truncated power series), but global solutions.

## 2. PRELIMINARIES AND DEFINITIONS

For further treatment on topics in this section see [6], [7], [8], [9], [10], and [14].

Definition 2.1. *['Galois Theory of Difference Equations' Example 1.3 [14]] Define* $S = \mathbb{C}^{\mathbb{N}}/\!\sim$ *where* $s_1 \sim s_2$ *if there exists* $N \in \mathbb{N}$ *such that, for all* $x > N$, $s_1(x) = s_2(x)$.

Lemma 2.2. *A unit is a sequence in* $S$ *that is invertible, i.e. a sequence that only has finitely many zeros.*

Definition 2.3. $V(L)$ *refers to the solution space of the operator* $L$, *i.e.* $V(L) = \{u \in S \mid Lu = 0\}$, *where* $S$ *is as in Definition 2.1.*

**Remark** $V(L)$ forms a subspace of $S$ over $\mathbb{C}$.

Theorem 2.4. *['A=B' Theorem 8.2.1 [13]]*
*Let* $L = \sum_{k=0}^{n} a_k \tau^k$ *be a linear difference operator of order* $n$ *on* $S$. *If* $a_0$ *and* $a_n$ *are units, then* $\dim(V(L)) = n$.

We view $\mathbb{C}(x)$ as a subset of $S$ (see Section 8.2 in [13]) so the theorem applies to $L \in \mathbb{C}(x)[\tau]$ with $a_0, a_n \neq 0$.

Definition 2.5. *Two operators* $L_1, L_2$ *in* $\mathbb{C}(x)[\tau]$ *are called* gauge equivalent *if and only if there exists an operator* $G \in \mathbb{C}(x)[\tau]$ *such that* $G(V(L_1)) = V(L_2)$ *and* $L_1, L_2$ *have the same order. This* $G$ *is a* gauge transformation $L_1 \to L_2$ *and we denote gauge equivalence by* $L_1 \sim_g L_2$.

If $L_1 \sim_g L_2$, and $G : V(L_1) \to V(L_2)$ is a gauge transformation, then the inverse gauge transformation $V(L_2) \to V(L_1)$ can be computed with the extended Euclidean Algorithm as follows: Let GCRD denote greatest common right divisor. Since $G : V(L_1) \to V(L_2)$ is a bijection, the kernel of $G$ on $V(L_1)$ is $\{0\}$ and thus $V(G) \cap V(L_1) = \{0\}$. So, $V(\mathrm{GCRD}(G, L_1)) = \{0\}$ and $\mathrm{GCRD}(G, L_1) = 1$. There exists $S, T \in \mathbb{C}(x)[\tau]$ such that $SG + TL_1 = 1$ by the extended Euclidean Algorithm for $\mathbb{C}(x)[\tau]$. Then $SG \equiv 1 \bmod L_1$. So $SG : V(L_1) \to V(L_1)$ is the identity map and $S : V(L_2) \to V(L_1)$ is the inverse of $G$.

Definition 2.6. *Let* $L_1, L_2 \in \mathbb{C}(x)[\tau]$. *The symmetric product of* $L_1$ *and* $L_2$ *written* $L_1 \otimes L_2$ *is defined as the monic operator* $L \in \mathbb{C}(x)[\tau]$ *of minimal order such that* $L(u_1u_2) = 0$ *for all* $u_1, u_2$ *with* $u_1 \in V(L_1)$ *and* $u_2 \in V(L_2)$. *For the case* $L_2 = \tau - r$ *with* $r \in \mathbb{C}(x)$ *we call* $\otimes L_2$ *a* term transformation *which is an action on* $\mathbb{C}(x)[\tau]$.

The formula for a term transformation is

$$L \otimes (\tau - r) = \frac{1}{b_n} \sum_{i=0}^{n} b_i \tau^i \qquad (1)$$
$$\text{where } b_n = a_n \text{ and } b_i(x) = a_i(x) \prod_{j=i}^{n-1} \tau^j(r(x)).$$

Given a series of gauge and term transformations from one operator to another, the following theorems reduce the problem of finding those transformations to that of finding exactly one gauge and one term transformation.

Theorem 2.7. *[Theorem 3.3 [8]] Let* $s_1, \ldots, s_m$ *be some combination of gauge transformations and term transformations. A transformation* $L_1 \xrightarrow{s_1 \circ \ldots \circ s_m} L_2$ *can be written* $L_1 \xrightarrow{t_2 \circ t_1} L_2$ *for some gauge transformation* $t_1$ *and some term transformation* $t_2$.

Definition 2.8. $L_1 \xrightarrow{t_2 \circ t_1} L_2$, *for some gauge transformation* $t_1$ *and some term transformation* $t_2$, *will be called a* GT-transformation.

Definition 2.9. *Let* $r(x) = cp_1(x)^{e_1} \cdots p_j(x)^{e_j} \in C(x)$ *with* $C \subseteq \mathbb{C}$. *Let the* $e_i \in \mathbb{Z}$, *let the* $p_i(x)$ *be irreducible in* $C[x]$, *and let* $s_i \in C$ *equal the sum of the roots of* $p_i(x)$. $r(x)$ *is said to be in* shift normal form *if* $-\deg(p_i(x)) < \mathrm{Re}(s_i) \leqslant 0$, *for* $i = 1, \ldots, j$. *We denote* $\mathrm{SNF}(r(x))$ *as the* shift normalized form *of* $r(x)$ *which is obtained by replacing each* $p_i(x)$ *by* $p_i(x + k_i)$ *for some* $k_i \in \mathbb{Z}$ *such that* $p_i(x + k_i)$ *is in shift normal form.*

**Remark:** If $r \in \mathbb{Q}(x)$ then factoring in $\mathbb{Q}[x]$ is easier than in $\mathbb{C}[x]$. Take $C = \mathbb{Q}$ in this case. Also, $\mathrm{SNF}(r(x))$ is unique up to choice of $C \subseteq \mathbb{C}$.

Definition 2.10. *Let* $L = a_2\tau^2 + a_1\tau + a_0$, $a_2 \neq 0$. *The* determinant *of* $L$, $\det(L)$, *is defined to be* $a_0/a_2$. *(It is the determinant of the 'companion matrix' in the corresponding matrix representation of* $L$.*)*

Theorem 2.11. *[Theorem 3.4 [8]] Let* $L_1, L_2$ *each have order* 2 *and the leading and trailing coefficient of* $L_1$ *be nonzero. If* $L_1 \xrightarrow{t_2 \circ t_1} L_2$ *for some gauge transformation* $t_1$ *and some term transformation* $t_2$ *then there exists a gauge transformation* $L_1 \otimes (\tau - r) \to L_2$, *where*

$$r = \pm\sqrt{\mathrm{SNF}(\det(L_2)/\det(L_1))}.$$

Definition 2.12. *It is in the context of Definition 2.8 (and Theorem 2.11) that we say that* $L_2$ *can be reduced to* $L_1$.

We provide an algorithm from [8] that finds such a reduction if it exists.

**Algorithm** Find GT-Transformation:
**Input:** $L_1, L_2 \in \mathbb{C}[x][\tau]$ linear difference operators of order 2.
**Output:** Operator of the form $H(x)(c_1(x)\tau + c_0(x))$ mapping $V(L_1)$ to $V(L_2)$ if it exists.

1. Calculate $\hat{r} = \mathrm{SNF}(\det(L_2)/\det(L_1))$.
2. If $\hat{r}$ is a square in $\mathbb{C}(x)$ then let $r = \sqrt{\hat{r}}$ else return 'FAIL' and stop.
3. Calculate $L_{neg} = L_1 \otimes (\tau - r)$ and $L_{pos} = L_1 \otimes (\tau + r)$.
4. Compute a gauge transformation, $c_1(x)\tau + c_0(x)$, between $L_{neg}$ and $L_2$ (see [3] or [8]).
   (a) If a gauge transformation exists then return $H(x) \cdot (c_1(x)\tau + c_0(x))$ and exit, where $H(x)$ is a solution of $\tau - r$.
   (b) If no gauge transformation exists then go to Step 5.
5. Compute a gauge transformation, $c_1(x)\tau + c_0(x)$, between $L_{pos}$ and $L_2$.

(a) If a gauge transformation exists then return $H(x)\cdot(c_1(x)\tau+c_0(x))$ and exit, where $H(x)$ is a solution of $\tau+r$.

(b) If no gauge transformation exists return 'FAIL.'

EXAMPLE 2.13. *Here we will check the above algorithm with operators in Example 6.2. Let* $L_1=-\frac{1}{3}(2+x)\tau^2+(2+\frac{4}{3}x)\tau-1-x$ *and* $L_2=(x+4)\tau^2+(-20-8x)\tau+(12x+12)$. *Then* $\hat{r}=4$ *and* $r=2$. *By computing gauge transformation from* $L_1\otimes(\tau-2)$ *to* $L_2$ *we get* $\frac{1}{x+2}(-\tau+3)$. *Thus the algorithm returns* $2^x(\frac{1}{x+2}(-2\tau+3))$. *(A 2 appeared in front of* $\tau$ *because* $\tau 2^x=2^x 2\tau$.*)*

# 3. FINITE SINGULARITIES

Local data of a difference operator are valuation growths at finite singularities in $\mathbb{C}/\mathbb{Z}$ and generalized exponents at the point of infinity. In this section, we will review the definition and an invariance property (Theorem 3.4) of valuation growths from [7], [9].

Let $L=a_n\tau^n+\cdots+a_0\tau^0\in\mathbb{C}(x)[\tau]$. After multiplying $L$ on the left by a suitable element of $\mathbb{C}(x)$, we may assume that the $a_i$ are in $\mathbb{C}[x]$ and $\gcd(a_0,\ldots,a_n)=1$.

DEFINITION 3.1. *Let* $L=a_n\tau^n+\cdots+a_0\tau^0$ *with* $a_i\in\mathbb{C}[x]$. $q\in\mathbb{C}$ *is called a* problem point *of* $L$ *if* $q$ *is a root of the polynomial* $a_0(x)a_n(x-n)$. $p\in\mathbb{C}/\mathbb{Z}$ *is called a* finite singularity *of* $L$ *if* $L$ *has a problem point in* $p$ *(i.e.* $p=q+\mathbb{Z}$ *for some problem point* $q$*).*

DEFINITION 3.2. *Let* $u(x)\in\mathbb{C}(x)$ *be a non-zero meromorphic function. The* valuation growth $g_p(u)$ *of* $u(x)$ *at* $p=q+\mathbb{Z}$ *is*

$$\liminf_{n\to\infty}(\text{order of } u(x) \text{ at } x=n+q)$$

$$-\liminf_{n\to\infty}(\text{order of } u(x) \text{ at } x=-n+q).$$

*Define the* set of valuation growths *of* $L$ *at* $p$ *as*

$$\overline{g}_p(L)=\{g_p(u)\mid u\neq 0 \text{ meromorphic solution of } L\}\subset\mathbb{Z}.$$

**Note**: The definition of valuation growths in [7] was longer, but using ideas from [2], the two definitions can be shown to be equivalent.

DEFINITION 3.3. *Let* $L$ *be a difference operator and* $p\in\mathbb{C}/\mathbb{Z}$ *be a finite singularity of* $L$. *If* $\overline{g}_p(L)$ *has more than one element then* $p$ *is called an* essential singularity.

THEOREM 3.4. *[Theorem 1 in [7]] If* $L_1$ *and* $L_2$ *are gauge equivalent then* $\overline{g}_p(L_1)=\overline{g}_p(L_2)$ *for every* $p\in\mathbb{C}/\mathbb{Z}$.

Let $\tilde{L}=L\otimes(\tau-a)$ for some $a\in\mathbb{C}(x)$ and let $\overline{g}_p(\tau-a)=\{d\}$ for some $d\in\mathbb{Z}$ and $p\in\mathbb{C}/\mathbb{Z}$. Then $\overline{g}_p(\tilde{L})=\{n+d\mid n\in\overline{g}_p(L)\}$. Therefore term transformations do not preserve $\overline{g}_p(L)$ but they do preserve $d_p(L):=\max\overline{g}_p(L)-\min\overline{g}_p(L)$. We define the set

$$\mathrm{Val}(L)=\{[p,d_p(L)]\mid p\in\mathbb{C}/\mathbb{Z} \text{ essential singularity of } L\}.$$

We compute $\mathrm{Val}(L)$ (see our table in Section 5) because it is data that is invariant under $GT$-transformations.

# 4. GENERALIZED EXPONENTS

Generalized exponents are local data at the point of infinity. A mathematically equivalent concept has been discussed in [4], [6], and [11]. The main techniques for computing generalized exponents are indicial equations, Newton $\tau$-polygons, and Newton $\Delta$-polygons of a difference operator, the same as in the computation of formal solutions in [4], [5]. Computing generalized exponents in an unramified case is also explained in [6, Section 5]. Here we will define generalized exponents of a difference operator and introduce Theorem 4.4, which is one of the key tools that the algorithm given in Section 7 uses.

Denote $t=1/x$ and let

$$K=\mathbb{C}((t)).$$

Define the following ring of difference operators:

$$\mathbf{D}:=K[\tau].$$

Now $\mathbb{C}(x)\subset K$ and the action of $\tau$ on $\mathbb{C}(x)$ can be extended to an action on $K$.

$$\tau(t)=\tau\left(\frac{1}{x}\right)=\frac{1}{x+1}=\frac{t}{1+t}=t-t^2+\cdots\in K.$$

The field $K$ has a natural valuation $v:K\to\mathbb{Z}\bigcup\{\infty\}$ where $v(0):=\infty$ and

$$v(c_nt^n+c_{n+1}t^{n+1}+\cdots)=n \quad \text{if } c_n\neq 0.$$

Let $\Delta=\tau-1$, then $\mathbf{D}=K[\tau]=K[\Delta]$. Let $L\in\mathbf{D}$ and write $L=\sum_{i=0}^{d}a_i\Delta^i$. Now we extend the definition of $v$ to $\mathbf{D}$ as follows

$$v(L):=\min\{v(a_i)+i\mid i=0,\ldots,d\}.$$

Note that this $v:\mathbf{D}\to\mathbb{Z}\bigcup\{\infty\}$ still satisfies the properties of a valuation:
(i) $v(L)=\infty\iff L=0$,
(ii) $v(L_1+L_2)\geq\min\{v(L_1),v(L_2)\}$, (equality when $v(L_1)\neq v(L_2)$)
(iii) $v(L_1L_2)=v(L_1)+v(L_2)$ (follows from Corollary 9.1 in the Appendix).

LEMMA 4.1. *Let* $L\in K[\tau]$. *There exists a polynomial* $P$ *such that for every* $n\in\mathbb{Z}$ *we have*

$$L(t^n)=P(n)t^{n+v(L)}+\cdots \tag{2}$$

*where the dots refer to terms of valuation* $>n+v(L)$.

PROOF. Let $\mathrm{tc}(f)$ be the trailing coefficient of $f\in\mathbb{C}((t))$. $\Delta^i(t^n)=P_i(n)t^{n+i}+\cdots$ where $P_i(n)=(-1)^in(n+1)\cdots(n+i-1)$ and $a_i\Delta^i(t^n)=P_i(n)\mathrm{tc}(a_i)t^{n+i+v(a_i)}+\cdots$. Let $M=\{i\in\mathbb{Z}\mid v(a_i)+i=v(L)\}$ then

$$L(t^n)=\sum_{i\in M}P_i(n)\mathrm{tc}(a_i)t^{n+v(L)}+\cdots.$$

Then $P(n)=\sum_{i\in M}P_i(n)\mathrm{tc}(a_i)$. $\square$

DEFINITION 4.2. $\mathrm{Ind}_L$, *the* indicial equation[2] *of* $L$, *is the polynomial,* $P(n)$, *constructed in the proof of lemma 4.1.*

[2] for futher discussion of the indicial equation, see the Appendix

For each $r \in \mathbb{N}$ we denote $K_r = \mathbb{C}((t^{1/r}))$. The algebraic closure of $K$ is $\overline{K} = \bigcup_{r\in\mathbb{N}} K_r$. Define the action of $\tau$ on $K_r$:

$$\begin{aligned}\tau(t^{\frac{1}{r}}) &= t^{\frac{1}{r}}(1+t)^{-\frac{1}{r}} \\ &= t^{\frac{1}{r}}(1-\frac{1}{1!}\frac{1}{r}t+\frac{1}{2!}\frac{1}{r}(\frac{1}{r}+1)t^2 \\ &\quad -\frac{1}{3!}\frac{1}{r}(\frac{1}{r}+1)(\frac{1}{r}+2)t^3+\cdots) \in K_r.\end{aligned} \tag{3}$$

Since we have defined the action of $\tau$ on $K_r$, we can now apply the formula for the term transformation in Equation (1) to $K_r[\tau]$. Let $\tilde{G}_r$ and $E_r$ be the following subgroup and subset, respectively, of $K_r^*$.

$$\tilde{G}_r = \left\{a \in K_r^* \mid a = 1+\sum_{i=r+1}^{\infty} a_i t^{i/r},\ a_i \in \mathbb{C}\right\}$$

$$E_r = \left\{a \mid a = ct^v(1+\sum_{i=1}^{r} a_i t^{i/r}), a_i \in \mathbb{C}, c \in \mathbb{C}^*, v \in \frac{1}{r}\mathbb{Z}\right\}$$

Now $E_r$ is a set of representatives for $K_r^*/\tilde{G}_r$. The composition of the natural maps $K_r^* \to K^*/\tilde{G}_r \to E_r$ defines a natural map

$$\mathrm{Trunc} : K_r^* \to E_r.$$

Let

$$G_r = \{a \in K_r^* \mid a = 1+\frac{m}{r}t+\sum_{i=r+1}^{\infty} a_i t^{i/r},\ a_i \in \mathbb{C},\ m \in \mathbb{Z}\}.$$

If $a, b \in E_r$ then we say $a \sim_r b$ when $a/b \in G_r$.
**Note:** $a \sim_r b$ if and only if $a_r \equiv b_r \bmod \frac{1}{r}\mathbb{Z}$ with $a_r$ as in the definition of $E_r$, and all the other parts of $a$ (the numbers $c, v, a_1, \ldots, a_{r-1}$) are the same for $b$.

Definition 4.3. *Let $a \in E_r$ for some $r \in \mathbb{N}$. We say that $a$ is a* generalized exponent *of $L$ with multiplicity $m$ if and only if 0 is a root of $Ind_{\tilde{L}}$ with multiplicity $m$ where $\tilde{L} = L \otimes (\tau - \frac{1}{a})$. We denote* $\mathrm{genexp}(L)$ *as the set of generalized exponents of $L$.*

Theorem 4.4. *Suppose $L_1 \sim_g L_2$, then for each $a \in$* $\mathrm{genexp}(L_1)$ *there exists $b \in$* $\mathrm{genexp}(L_2)$ *such that $a \sim_r b$. (Where $r$ is minimal with $a \in E_r$.)*

Proof. Let $G \in \mathbb{C}(x)[\tau]$ be a gauge transformation from $L_1$ to $L_2$ and $a \in E_r$ for some $r \in \mathbb{N}$ be a generalized exponent of $L_1$. One can verify that $G_A := A \cdot G \cdot A^{-1}$ is an element of $\mathbb{C}(x,a)[\tau]$ and $G_A$ is a bijection from $V(L_1 \otimes (\tau - 1/a))$ to $V(L_2 \otimes (\tau - 1/a))$ where $A$ is a solution of $\tau - 1/a$ (a basis of solutions of any operator in $\overline{K}[\tau]$ can be found in the universal extension of $\overline{K}$, see Section 6.2 of [14]). Since $a$ is a generalized exponent of $L_1$, $L_1 \otimes (\tau - 1/a)$ has 0 as a root of its indicial equation. Then it has a solution in $K_r$ by Lemma 9.3 in the Appendix. By applying $G_A$, we find that $L_2 \otimes (\tau - 1/a)$ also has a solution in $K_r$. Then its indicial equation has a root in $\frac{1}{r}\mathbb{Z}$ by Lemma 9.3 in the Appendix. Let $m/r$ be such a root for some $m \in \mathbb{Z}$. Then

$$L_2 \otimes (\tau - 1/a) \otimes (\tau - (1+\frac{m}{r}t))$$

has 0 as a root of the indicial equation. Then by Equation 1, $(\tau - 1/a) \otimes (\tau - (1+\frac{m}{r}t)) = \tau - \frac{1}{a}(1+\frac{m}{r}t)$. Thus, $\mathrm{Trunc}(a/(1+\frac{m}{r}t))$ is a generalized exponent of $L_2$. $\square$

The above theorem says generalized exponents mod $\sim_r$ are invariant under gauge transformations. Suppose $L$ has order 2, and let $\mathrm{genexp}(L) = \{a_1, a_2\}$ and $\tilde{L} = L \otimes (\tau - \alpha)$ for some $\alpha \in K_r, r \in \mathbb{N}$. Then

$$\mathrm{genexp}(\tilde{L}) = \{\mathrm{Trunc}(\alpha a_1), \mathrm{Trunc}(\alpha a_2)\}.$$

To obtain an expression that is invariant under the term transformations as well, we define the quotient of the generalized exponents.

Definition 4.5. *Suppose $L$ has order 2, and* $\mathrm{genexp}(L) = \{a_1, a_2\}$ *such that $v(a_1) \geq v(a_2)$. If $v(a_1) > v(a_2)$ then we define the* set of quotient of the two generalized exponents *as* $\mathrm{Gquo} = \{\mathrm{Trunc}(a_1/a_2)\}$. *If $v(a_1) = v(a_2)$ then we define* $\mathrm{Gquo}(L) = \{\mathrm{Trunc}(a_1/a_2), \mathrm{Trunc}(a_2/a_1)\}$.

An example of computing a generalized exponent in unramified case is explained in [6]. An example follows:

Example 4.6. *$L_{WM} = (2n+2\nu+3+2x)\tau^2 + (2z-4\nu-4x-4)\tau - 2n+1+2\nu+2x$ is an operator from the table in Section 5. Suppose* $\mathrm{genexp}(L_{WM}) = \{c_1 t_1^{v}(1+a_1 t^{\frac{1}{2}}+a_2 t), c_2 t_2^{v}(1+b_1 t^{\frac{1}{2}}+b_2 t)\}$. *The slope of the Newton $\tau$-polygon of $L_{WM}$ is 0 and the corresponding Newton $\tau$-polynomial is $2(t-1)^2$. So $c_1 = c_2 = 1$ and $v_1 = v_2 = 0$ (the $c_i, v_i$ are the values for the numbers $c, v$ from the definition of $E_r$). Since* $\mathrm{Ind}_{L_{WM}} = 2z$, $\mathrm{Ind}_{L_{WM}}$ *has no root, that is, $a_1$ and $b_1$ are not 0. So we need to calculate the Newton $\delta$-polygon of $L_{WM}$. Then the slope of the Newton $\delta$-polygon is $\frac{1}{2}$ and its Newton $\delta$-polynomial is $2z+2t^2$, which gives $a_1 = \sqrt{-z}$ and $b_1 = -\sqrt{-z}$. The indicial equation of both $L_{WM} \otimes (\tau - 1/(\sqrt{-z}t^{\frac{1}{2}}))$ and $L_{WM} \otimes (\tau - (1/-\sqrt{-z}t^{\frac{1}{2}}))$ is $-64z(1+2z+4\nu)+256nz$, so the root of the indicial equation is $n = \frac{1}{4}+\frac{1}{2}z+\nu$. Thus, $a_2 = b_2 = -n$,* $\mathrm{genexp}(L_{WM}) = \{1+\sqrt{-z}t^{\frac{1}{2}}-(\frac{1}{4}+\frac{1}{2}z+\nu)t, 1-\sqrt{-z}t^{\frac{1}{2}}-(\frac{1}{4}+\frac{1}{2}z+\nu)t\}$, *and* $\mathrm{Gquo} = \{1-2\sqrt{-z}t^{\frac{1}{2}}-2zt, 1+2\sqrt{-z}t^{\frac{1}{2}}-2zt\}$.

# 5. BASE EQUATIONS OF DEGREE OF 2

Many special functions satisfy recurrences w.r.t their parameters as in [1]. We use these recurrences as base equations in the table below. The table also contains the corresponding local data that we computed with our implementation. At the moment the table below contains a somewhat arbitrary set of base equations. It is easy to add more, and our goal is to do that in a systematic way. The table lists base equations with their known solution(s) and local data. Here $B_I$ and $B_J$ denote Bessel functions of the first kind, $B_K$ and $B_Y$ denote Bessel functions of the second kind, $W_W$ denotes the Whittaker $W$ function and $W_M$ denotes the Whittaker $M$ function.

1. $L_{IK} = z\tau^2 + (2+2\nu+2x)\tau - z$
   - Constants: $z, \nu$
   - Solutions $= \{B_I(\nu+x, z), B_K(\nu+x, -z)\}$
   - Gquo $= \{-\frac{z^2}{4}t^2(1+(-1-2\nu)t)\}$
   - Val $= \{\}$
2. $L_{JY} = z\tau^2 - (2+2\nu+2x)\tau + z$
   - Constants: $z, \nu$
   - Solutions $= \{B_J(\nu+x, z), B_Y(\nu+x, z)\}$

- Gquo $= \{\frac{z^2}{4}t^2(1+(-1-2\nu)t)\}$
- Val $= \{\}$

3. $L_{WW} = \tau^2 + (z-2\nu-2x-2)\tau - \nu - x - \frac{1}{4} - \nu^2 - 2\nu x - x^2 + \eta^2$
   - Constants: $z, \nu, \eta$
   - Solution $= \{W_W(\nu+x, \eta, z)\}$
   - Gquo $= \{(-3+2\sqrt{2})(1+\frac{1}{2}\sqrt{2}zt), (-3-2\sqrt{2})(1-\frac{1}{2}\sqrt{2}zt)\}$
   - Val $= \{[\eta+\frac{1}{2}-\nu, 1], [-\eta+\frac{1}{2}-\nu, 1]\}$ if $\eta \notin \frac{1}{2}\mathbb{Z}$ or Val $= \{[\eta+\frac{1}{2}-\nu, 2]\}$ if $\eta \in \frac{1}{2}+\mathbb{Z}$

4. $L_{WM} = (2\eta+2\nu+3+2x)\tau^2 + (2z-4\nu-4x-4)\tau - 2\eta + 1 + 2\nu + 2x$
   - Constants: $z, \nu, \eta$
   - Solution $= \{W_M(\nu+x, \eta, z)\}$
   - Gquo $= \{1-2\sqrt{-z}t^{\frac{1}{2}} - 2zt, 1+2\sqrt{-z}t^{\frac{1}{2}} - 2zt\}$
   - Val $= \{[\eta+\frac{1}{2}-\nu, 1], [-\eta+\frac{1}{2}-\nu, 1]\}$ if $\eta \notin \frac{1}{2}\mathbb{Z}$ or Val $= \{[\eta+\frac{1}{2}-\nu, 2]\}$ if $\eta \in \frac{1}{2}+\mathbb{Z}$

5. $L_{{}_2F_1} = (z-1)(a+x+1)\tau^2 + (-z+2-za-zx+2a+2x+zb-c)\tau - a + c - 1 - x$
   - Constants: $a, b, c, z$
   - Solution $= \{{}_2F_1(a+x, b; c; z)\}$
   - Gquo $= \{(1-z)(1+(c-2b)t), \frac{1}{1-z}(1+(2b-c)t)\}$
   - Val $= \{[-a, 1], [-a+c, 1]\}$ if $c \notin \mathbb{Z}$ or Val $= \{[-a, 2]\}$ if $c \in \mathbb{Z}$

In case 5, whenever $b \in [0, -1, -2, \ldots]$, ${}_2F_1(a+x, b; c; z)$ satisfies a first order recurrence equation as mentioned in [8, Remark 4.1]. So, this case is not of interest to this algorithm. Also, $u(x) = \frac{\Gamma(a+x+1-c)}{\Gamma(a+x)}\ {}_2F_1(a+x+1-c, b+1-c; 2-c; z)$ is another solution of $L_{{}_2F_1}$ when $u(x)$ is defined and $c \notin \mathbb{Z}$ by [8, Theorem 4.4].

# 6. EXAMPLES

Example 6.1. *Sequence A096121 = $[2, 8, 60, 816, 17520, 550080, \ldots]$ in [12] represents the "Number of full spectrum rook's walks on a $(2 \times n)$ board" and it is a solution of the recurrence operator $A = \tau^2 - (x+1)(x+2)\tau - (x+1)(x+2)$. The local data of $A$ is*

$$\mathrm{Gquo}(A) = \{-t^2(1-t)\} \quad and \quad \mathrm{Val} = \{\}.$$

*The local data of $A$ matches the operator $L_{IK}$ in the table in Section 5. Before we can call algorithm Find GT-transformation we need to find explicit values for the unknown constants $z$ and $\nu$ appearing in $L_{IK}$. Since $\tau(B_J(\nu+x, z)) = B_J(\nu+x+1, z)$ and $\tau$ is a gauge transformation, we only need $\nu$ mod $\mathbb{Z}$. Comparing* Gquo$(A)$ *with* Gquo$(L_{IK})$ *(see table in Section 5) using Theorem 4.4 gives $-1 \equiv -1-2\nu$ mod $\mathbb{Z}$ and $-\frac{z^2}{4} = -1$. Hence $\nu \in \frac{1}{2}+\mathbb{Z}$ or $\nu \in 0+\mathbb{Z}$ and $z = \pm 2$. So, if $A$ can be reduced to $L_{IK}$ for some parameter value, then $A$ can be reduced to one of:*

$$-2\tau^2 + (2+2x)\tau + 2, \quad -2\tau^2 + (3+2x)\tau + 2,$$

$$2\tau^2 + (2+2x)\tau - 2, \quad 2\tau^2 + (3+2x)\tau - 2.$$

*(These are $L_{IK}$ with $\nu \in \{0, \frac{1}{2}\}, z \in \{2, -2\}$.) Then algorithm Find GT-transformation finds that $A$ can be reduced to $-2\tau^2 + (2+2x)\tau + 2$. It also finds the gauge transformation 1 and the term product $\tau - (x+1)$. From the list, a basis of solutions of $-2\tau^2 + (2+2x)\tau + 2$ is*

$$\{B_I(x, -2), B_K(x, 2)\}.$$

*By applying the gauge transformation and the term product we get a basis of solutions of $A$ as*

$$\{B_I(x, -2)\Gamma(x+1), B_K(x, 2)\Gamma(x+1)\}.$$

.

Example 6.2. *Sequence A005572 = $[1, 4, 17, 76, 354, 1704, 8421, \ldots]$ in [12] represents the "Number of walks on cubic lattice starting and finishing on the xy-plane and never going below it" and it is a solution of the recurrence operator $H = (x+4)\tau^2 + (-20-8x)\tau + (12x+12)$. This same example has been used in [8] also. The local data of $H$ is*

$$\mathrm{Gquo}(A) = \{\frac{1}{3}, 3\} \quad and \quad \mathrm{Val} = \{[0, 2]\}.$$

*The local data of $H$ matches with the operator $L_{{}_2F_1}$ in the table in Section 5. Since* Val $= \{[0, 2]\}$, *we get $a, c \in 0+\mathbb{Z}$. We may take $a = 1$ and $c = 1$ so that ${}_2F_1$ is defined. Comparing* Gquo *gives $c-2b \equiv 0$ mod $\mathbb{Z}$ and $1-z \in \{1/3, 3\}$. By Lemma 4.3 in [8] we need $b$ mod $\mathbb{Z}$, so $b \in 0+\mathbb{Z}$ or $b \in \frac{1}{2}+\mathbb{Z}$. So, if $H$ can be reduced to $L_{{}_2F_1}$ for some parameter values, then $H$ can be reduced to one of:*

$$-3(2+x)\tau^2 + (7+4x)\tau - 1 - x, \quad -3(2+x)\tau^2 + (6+4x)\tau - 1 - x$$

$$\frac{-1}{3}(2+x)\tau^2 + (\frac{5}{3}+\frac{4}{3}x)\tau - 1 - x, \quad \frac{-1}{3}(2+x)\tau^2 + (2+\frac{4}{3}x)\tau - 1 - x.$$

*(These are $L_{{}_2F_1}$ with $a = 1, b \in \{0, \frac{1}{2}\}, c = 1, z \in \{-2, \frac{2}{3}\}$.) Then algorithm Find GT-transformation finds that $H$ can be reduced to $-\frac{1}{3}(2+x)\tau^2 + (2+\frac{4}{3}x)\tau - 1 - x$ with gauge transformation $\frac{1}{x+2}(-\tau+3)$ and term product $\tau - 2$. From the table, a solution of $-\frac{1}{3}(2+x)\tau^2 + (2+\frac{4}{3}x)\tau - 1 - x$ is*

$${}_2F_1(x+1, \frac{1}{2}; 1; \frac{2}{3}).$$

*By applying the gauge transformation and the term product we get a solution of $A$ and after checking initial values, we find that the sequence equals*

$$\frac{2^{x+1}}{\sqrt{3}(x+2)}\left({}_2F_1(x+2, \frac{1}{2}; 1; \frac{2}{3}) \cdot 2 - {}_2F_1(x+1, \frac{1}{2}; 1; \frac{2}{3}) \cdot 3\right).$$

# 7. ALGORITHM

As in the examples in Section 6, our algorithm will compute a number of candidates (in Step 3), and then try to reduce the input equation to one of those candidates (in Step 4). Here is how the list of candidates given in Step (3h) were determined (the other cases in Step 3 can be done in the same way). Suppose we have an operator $L$ that has $c_1(1+d_2t) \in \mathrm{Gquo}(L)$ and $\mathrm{Val}(L) = \{[m, 2]\}$ where $c_1 \geq 1, d_2 \in \mathbb{C}$, and $m \in \mathbb{Z}$. Then $L$ matches the operator $L_{{}_2F_1} = (z-1)(a+x+1)\tau^2 + (-z+2-za-zx+2a+2x+zb-c)\tau - a + c - 1 - x$ in Section 5. If $L$ can be reduced to an operator $L_{{}_2F_1}$ then $c_1$ should equal either $1-z$ or $\frac{1}{1-z}$. From $\mathrm{Val}(L)$ we get $a = -m$ and $c \in \mathbb{Z}$. Since we need

$c \bmod \mathbb{Z}$ [8, Lemma 4.3], we may let $c = 0$. Also, we need $c - 2b$ or $2b - c \bmod \mathbb{Z}$, so candidates for $b$ are $-\frac{d_2}{2}$ and $-\frac{d_2+1}{2}$. Hence candidates for the parameters $[a, b, c, z]$ are $[-m, -\frac{d_2}{2}, 0, 1-c_1], [-m, -\frac{d_2+1}{2}, 0, 1-c_1], [-m, -\frac{d_2}{2}, 0, 1-\frac{1}{c_1}], [-m, -\frac{d_2+1}{2}, 0, 1-\frac{1}{c_1}]$. The other cases in Step 3 of the following algorithm were generated similarly.

**Algorithm**: solver
**Input**: An operator $L = a_2\tau^2 + a_1\tau + a_0 \in \mathbb{Q}(x)[\tau]$.
**Output**: At least one solution of $L$ if there is an operator in the table in Section 5 to which $L$ can be reduced. Otherwise the empty set.

1. $comb := \emptyset, const := \emptyset$
2. Calculate $\mathrm{Gquo}(L)$ and $\mathrm{Val}(L)$.
3. For $f \in \mathrm{Gquo}(L)$ let $f = c_1 t^v(1 + d_1 t^{1/2} + d_2 t)$.
   (a) If $v = 2$, $c_1 > 0$, $d_1 = 0$, and $\mathrm{Val} = \{\}$ then
      i. Let $Z := \{2\sqrt{c_1}, -2\sqrt{c_1}\}, V := \{-\frac{1}{2}d_2 - \frac{1}{2}, -\frac{1}{2}(d_2+1) - \frac{1}{2}\}$
      ii. $comb := \{z\tau^2 - (2+2v+2x)\tau + z \mid v \in V, z \in Z\}$
   (b) If $v = 2$, $c_1 < 0$, $d_1 = 0$, and $\mathrm{Val} = \{\}$ then
      i. Let $Z := \{2\sqrt{-c_1}, -2\sqrt{-c_1}\}, V := \{-\frac{1}{2}d_2 - \frac{1}{2}, -\frac{1}{2}(d_2+1) - \frac{1}{2}\}$
      ii. $comb := \{z\tau^2 - (2+2v+2x)\tau + z \mid v \in V, z \in Z\}$
   (c) If $v = 0, d_1 = 0, \mathrm{Val}(L) = \{[m, 2]\}$, and $c_1 = -3 + 2\sqrt{2}$ then
      i. Let $NV := \{[0, \frac{1}{2} - m], [\frac{1}{2}, -m]\}$,
      $z :=$ rational part of $\sqrt{2}d$
      ii. $comb := \{\tau^2 + (z - 2\nu - 2x - 2)\tau - \nu - x - \frac{1}{4} - \nu^2 - 2\nu x - x^2 + n^2 \mid [n, \nu] \in NV\}$
   (d) If $v = 0, d_1 = 0, \mathrm{Val}(L) = \{[n_1, 1], [n_2, 1]\}$, and $c_1 = -3 + 2\sqrt{2}$ then
      i. Let $NV := \{[-\frac{n_1+n_2}{2}, \frac{n_1-n_2-1}{2}],$
      $[-\frac{n_1+n_2}{2}, \frac{-n_1+n_2-1}{2}], [-\frac{n_1+n_2+1}{2}, \frac{n_1-n_2}{2}],$
      $[-\frac{n_1+n_2+1}{2}, \frac{-n_1+n_2}{2}]\}$,
      $z :=$ rational part of $\sqrt{2}d$
      ii. $comb := \{\tau^2 + (z - 2\nu - 2x - 2)\tau - \nu - x - \frac{1}{4} - \nu^2 - 2\nu x - x^2 + n^2 \mid [n, \nu] \in NV\}$
   (e) If $v = 0, d_1 \neq 0, \mathrm{Val}(L) = \{[m, 2]\}$, and $c_1 = 1$ then
      i. Let $NV := \{[0, \frac{1}{2} - m], [\frac{1}{2}, -m]\}, z := -\frac{d_1^2}{4}$
      ii. $comb := \{\tau^2(2n + 2\nu + 3 + 2x) + (2z - 4\nu - 4x - 4)\tau - 2n + 1 + 2\nu + 2x \mid [n, \nu] \in NV\}$
   (f) If $v = 0, d_1 = 0, \mathrm{Val}(L) = \{[n_1, 1], [n_2, 1]\}$, and $c_1 = -3 + 2\sqrt{2}$ then
      i. Let $NV := \{[-\frac{n_1+n_2}{2}, \frac{n_1-n_2-1}{2}],$
      $[-\frac{n_1+n_2}{2}, \frac{-n_1+n_2-1}{2}], [-\frac{n_1+n_2+1}{2}, \frac{n_1-n_2}{2}],$
      $[-\frac{n_1+n_2+1}{2}, \frac{-n_1+n_2}{2}]\}$,
      $z := -\frac{d_1^2}{4}$
      ii. $comb := \{\tau^2(2n + 2\nu + 3 + 2x) + (2z - 4\nu - 4x - 4)\tau - 2n + 1 + 2\nu + 2x \mid [n, \nu] \in NV\}$
   (g) If $v = 0, d_1 = 0, c_1 \geq 1$, and $\mathrm{Val}(L) = \{[m, 2]\}$ then
      i. Let $const := \{[-m, -\frac{d_2}{2}, 0, 1 - c_1],$
      $[-m, -\frac{d_2+1}{2}, 0, 1 - c_1], [-m, -\frac{d_2}{2}, 0, 1 - \frac{1}{c_1}],$
      $[-m, -\frac{d_2+1}{2}, 0, 1 - \frac{1}{c_1}]\}$
      ii. $comb := \{(z-1)(a+x+1)\tau^2 + (-z + 2 - za - zx + 2a + 2x + zb - c)\tau - a + c - 1 - x \mid [a, b, c, z] \in const\}$
   (h) If $v = 0$, $d_1 = 0$, $c_1 \geq 1$, and $\mathrm{Val}(L) = \{[n_1, 1], [n_2, 1]\}$ then
      i. Let $const := \{[-n_1, -\frac{n_2-n_1-d_2}{2}, n_2 - n_1, 1 - c_1],$
      $[-n_1, -\frac{n_2-n_1-d_2+1}{2}, n_2 - n_1, 1 - c_1],$
      $[-n_1, -\frac{n_2-n_1-d_2}{2}, n_2 - n_1, 1 - \frac{1}{c_1}],$
      $[-n_1, -\frac{n_2-n_1-d_2+1}{2}, n_2 - n_1, 1 - \frac{1}{c_1}],$
      $[-n_2, -\frac{n_1-n_2-d_2}{2}, n_1 - n_2, 1 - c_1],$
      $[-n_2, -\frac{n_1-n_2-d_2+1}{2}, n_1 - n_2, 1 - c_1],$
      $[-n_2, -\frac{n_1-n_2-d_2}{2}, n_1 - n_2, 1 - \frac{1}{c_1}],$
      $[-n_2, -\frac{n_1-n_2-d_2+1}{2}, n_1 - n_2, 1 - \frac{1}{c_1}]\}$
      ii. For each $[a, b, c, z] \in const$, if ${}_2F_1(a, b; c; z)$ is not defined, then shift $a, b, c$ by a suitable integer (this changes ${}_2F_1$ only by a gauge transformation [8, Lemma 4.3]).
      iii. $comb := \{(z-1)(a+x+1)\tau^2 + (-z + 2 - za - zx + 2a + 2x + zb - c)\tau - a + c - 1 - x \mid [a, b, c, z] \in const\}$
   (i) Otherwise stop and return $\emptyset$.
4. For each $L_c \in comb$ check if $L$ can be reduced to $L_c$ and if so
   (a) Generate a basis of solutions or a solution of $L_c$ by plugging in corresponding parameters.
   (b) Apply the term transformation and gauge transformation to the result from Step (4a).
   (c) Return the result of Step (4b) as output and stop the algorithm.

# 8. IMPROVEMENTS

From [12], we can obtain a large number of equations for which useful things are known, such as references, formulas, etc. These equations can be added to the table. Since they do not contain parameters, the key problem solved in this paper (finding parameter values) becomes empty, and is replaced with a new problem: how to quickly select the right equation from a large collection? This problem was solved in G. Levy's Ph.D thesis [8] using the $p$-curvature. Also treated in [8] was the reduction to $L_{{}_2F_1}$, and to $\tau^2 + b_0$ (Liouvillian solutions). The thesis and implementation can be copied from [8].

## 8.1 Future work

The table in Section 5 contains a small number of base equations. We want to extend this table significantly. Given an equation and solution(s), it is easy to add the equation to the table; all we have to do is to compute its local data with our implementation and compute formulas for the parameters from that. We will need to set up a systematic approach to construct base equations and their solutions to ensure that none will be overlooked. Also for higher order, we can apply the same techniques.

# 9. APPENDIX

In this section, we will discuss more about indicial equation of a linear difference operator and state Lemma 9.3, which has been used to prove Theorem 4.4.

Let $u \in K$, $u \neq 0$, and $v(u) = n$. Write

$$u = c_n t^n + c_{n+1} t^{n+1} + \cdots$$

then it follows from equation (2) (use the fact that $L(a+b) = L(a) + L(b)$) that

$$L(u) = c_n P(n) t^{n+v(L)} + \cdots$$

Corollary 9.1. *Let $u \in K$, $u \neq 0$, then*

$$v(L(u)) = v(u) + v(L) \Longleftrightarrow v(u) \text{ is not a root of } \mathrm{Ind}_L$$

Lemma 9.2. *Let $L \in K[\tau]$ and $L \neq 0$. Then*

$$\dim(\mathrm{Ker}(L, K)) > 0 \Longleftrightarrow \mathrm{mult}_{\mathbb{Z}}(\mathrm{Ind}_L) > 0$$

*where* $\mathrm{Ind}_L$ *is the indicial equation of $L$, and* $\mathrm{mult}_{\mathbb{Z}}(\mathrm{Ind}_L)$ *denotes the number of integer roots of* $\mathrm{Ind}_L$.

Proof. "$\Longrightarrow$" if $u \in K$, $u \neq 0$, and $L(u) = 0$ then $v(u)$ must be a root of $\mathrm{Ind}_L$ by Corollary 9.1.
"$\Longleftarrow$" Let $n$ be the *largest* integer root of $\mathrm{Ind}_L$, so

$$\mathrm{Ind}_L(n) = 0,\ \mathrm{Ind}_L(n+1) \neq 0,\ \mathrm{Ind}_L(n+2) \neq 0, \ldots \quad (4)$$

Since $\mathrm{Ind}_L(n) = 0$ it follows from equation (2) that

$$L(t^n) = t^{n+v(L)} \cdot (0t^0 + a_1 t^1 + a_2 t^2 + \cdots)$$

Write

$$u = t^n + c_{n+1} t^{n+1} + c_{n+2} t^{n+2} + \cdots$$

Then write

$$L(u) = t^{n+v(L)} \cdot (0t^0 + A_1 t^1 + A_2 t^2 + \cdots)$$

Now $A_1 = a_1 + c_{n+1}\mathrm{Ind}_L(n+1)$ and since $\mathrm{Ind}_L(n+1) \neq 0$ there is a unique $c_{n+1} \in \mathbb{C}$ for which $A_1$ vanishes, namely $c_{n+1} := -a_1/\mathrm{Ind}_L(n+1)$. Then $A_2$ equals some constant plus $c_{n+2}\mathrm{Ind}_L(n+2)$, and again $\mathrm{Ind}_L(n+2) \neq 0$ so there is a unique $c_{n+2}$ for which $A_2$ vanishes. Continuing this way leads to $L(u) = 0$. $\square$

We can say the same for $L \in K_r[\tau]$ for $r \in \mathbb{N}$.

Lemma 9.3. *Let $L \in K[\tau]$ and $L \neq 0$. Then*

$$\dim(\mathrm{Ker}(L, K_r)) > 0 \Longleftrightarrow \mathrm{mult}_{\frac{1}{r}\mathbb{Z}}(\mathrm{Ind}_L) > 0$$

*where* $\mathrm{Ind}_L$ *is the indicial equation of $L$, and* $\mathrm{mult}_{\frac{1}{r}\mathbb{Z}}(\mathrm{Ind}_L)$ *denotes the number of roots of* $\mathrm{Ind}_L$ *in* $\frac{1}{r}\mathbb{Z}$ .

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