Liouvillian Solutions of Irreducible Second Order Linear Difference Equations

Mark van Hoeij and Giles Levy

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Preliminaries

Definition

τ will refer to the shift operator acting on $\mathbb{C}(n)$ by $\tau : n \mapsto n+1$.

An operator $L = \sum_i a_i \tau^i$ acts as $Lu(n) = \sum_i a_i u(n+i)$.

Definition

$\mathbb{C}(n)[\tau]$ is the ring of *linear difference operators* where ring multiplication is composition of operators $L_1 L_2 = L_1 \circ L_2$.

Definition

Let $S = \mathbb{C}^{\mathbb{N}}/\sim$ where $s_1 \sim s_2$ if there exists $N \in \mathbb{N}$ such that, for all $n > N$, $s_1(n) = s_2(n)$.

Definition

$V(L)$ refers to the solution space of the operator L, i.e.
$V(L) := \{u \in S \mid Lu = 0\}$.

If $L = \sum_{i=0}^{k} a_i \tau^i, \ a_0, a_k \neq 0$, then $\dim(V(L)) = k$
('A=B' Theorem 8.2.1).

Definition

A function or sequence $v(n)$ such that $v(n+1)/v(n) = r(n)$ is a rational function of n will be called a *hypergeometric term*.

Tools

Let $D = \mathbb{C}(n)[\tau]$. If $L \in D$ with $L \neq 0$ then D/DL is a $D-$module.

Definition

L_1 is *gauge equivalent* to L_2 when D/DL_1 and D/DL_2 are isomorphic as $D-$modules.

Lemma

L_1 is gauge equivalent to L_2 if and only if $\exists$ $G \in D$ such that $G(V(L_1)) = V(L_2)$ and L_1, L_2 have the same order. Thus G defines a bijection $V(L_1) \to V(L_2)$.

Definition

The bijection defined by G in the preceding lemma will be called a *gauge transformation*.

Definition

The *companion matrix* of a monic difference operator

$$L = \tau^k + a_{k-1}\tau^{k-1} + \cdots + a_0, \;\; a_i \in \mathbb{C}(n)$$

will refer to the matrix:

$$M = \begin{pmatrix} 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & \ldots & -a_{k-2} & -a_{k-1} \end{pmatrix}.$$

The equation $Lu = 0$ is equivalent to the system $\tau(Y) = MY$ where

$$Y = \begin{pmatrix} u(n) \\ \vdots \\ u(n+k-1) \end{pmatrix}.$$

Definition

Let $L = a_k\tau^k + a_{k-1}\tau^{k-1} + \cdots + a_0,\ a_i \in \mathbb{C}(n)$. The *determinant* of L, $\det(L) := (-1)^k a_0/a_k$, i.e. the determinant of its companion matrix.

Definition

Two rational functions will be called *shift equivalent*, denoted $r_1 \overset{\mathrm{SE}}{\equiv} r_2$, if

- $\tau - r_1/r_2$ has a rational solution

or, equivalently,

- the difference modules for $\tau - r_1$ and $\tau - r_2$ are isomorphic.

Lemma

If there exists a gauge transformation $G : V(L_1) \to V(L_2)$ then $\det(L_1) \overset{\mathrm{SE}}{\equiv} \det(L_2)$.

Liouvillian

Liouvillian solutions are defined in Hendriks-Singer 1999 Section 3.2. For irreducible operators they are characterized by the following theorem:

Theorem (Propositions 31-32 in Feng-Singer-Wu 2009 or Lemma 4.1 in Hendriks-Singer 1999)

An irreducible k'th order operator L has Liouvillian *solutions if and only if L is gauge equivalent to* $\tau^k + \alpha,\ \alpha \in \mathbb{C}(n)$.

Finding a gauge equivalence to $\tau^k + \alpha$ is desirable because it is easily solved with interlaced hypergeometric terms, e.g. $\tau^2 - 4(n+2)/(n+7)$ has solutions:

$$\frac{\Gamma(\frac{n}{2}+1)}{\Gamma(\frac{n}{2}+\frac{7}{2})} \cdot 2^n \cdot \begin{cases} k_1, & \text{if } n \text{ even} \\ k_2, & \text{if } n \text{ odd} \end{cases}$$

where k_1, k_2 are arbitrary constants.

Definition

Let $L_1, L_2 \in \mathbb{C}(n)[\tau]$. The *symmetric product* of L_1 and L_2 is defined as the monic operator $L \in \mathbb{C}(n)[\tau]$ of smallest order such that $L(u_1 u_2) = 0$ for all $u_1, u_2 \in S$ with $L_1 u_1 = 0$ and $L_2 u_2 = 0$.

Definition

The *symmetric square* of L, denoted $L^{\circledS 2}$, will refer to the symmetric product of L and L (i.e. with itself).

Lemma

Let $L = a_2\tau^2 + a_1\tau + a_0,\ \ a_0, a_2 \neq 0.$

$$L^{\circledS 2} \text{ has order: } \begin{cases} 2, & \text{if } a_1 = 0 \\ 3, & \text{if } a_1 \neq 0 \end{cases}$$

Commutative Diagram

$$L = a_2\tau^2 + a_1\tau + a_0, \quad a_1 \neq 0$$
$$\tilde{L} = \tau^2 + \alpha$$
$$G = \tau + g$$
$$\alpha, g \in \mathbb{C}(n), \ \textit{unknown}$$

$$\begin{array}{ccccccc}
 & & V(L) & \xrightarrow{\quad G \quad} & V(\tilde{L}) & \longrightarrow & 0 \\
 & & \Big\downarrow {\scriptstyle u \downarrow u^2} & & \Big\downarrow {\scriptstyle v \downarrow v^2} & & \\
0 \to V(GCRD(G_2, L^{\circledS 2})) & \to & V(L^{\circledS 2}) & \xrightarrow{\quad G_2 \quad} & V(\tilde{L}^{\circledS 2}) & \longrightarrow & 0 \\
\text{dim } 1 & & \text{dim } 3 & & \text{dim } 2 & &
\end{array}$$

Algorithm

Algorithm Find Liouvillian:

Input: $L \in \mathbb{C}[n][\tau]$ a second order, irreducible, homogeneous difference operator.

Let $L = a_2(n)\tau^2 + a_1(n)\tau + a_0(n)$ and let $L^{Ⓢ2} = c_3\tau^3 + c_2\tau^2 + c_1\tau + c_0$.

Output: A two-term difference operator, $\hat{L}$, with a gauge transformation from $\hat{L}$ to L, if it exists.

1. If $a_1 = 0$ then return $\hat{L} = L$ and stop.
2. Let $u(n)$ be an indeterminate function. Impose the relation $Lu(n) = 0$, i.e.

$$u(n+2) = -\frac{1}{a_2(n)}(a_0(n)u(n) + a_1(n)u(n+1)).$$

Algorithm (continued)

3. Let $d = \det(L) = a_0/a_2$. Let R be a non-zero rational solution of
$$L_T := L^{Ⓢ2} \otimes (\tau + 1/d),$$
if such a solution exists, else return NULL and stop.
4. Let g be an indeterminate and let
$$G := \tau + g : V(L) \longrightarrow V(\hat{L})$$
Compute corresponding $G_2 : V(L^{Ⓢ2}) \to V(\hat{L}^{Ⓢ2})$.
5. From R (solution of L_T) take the corresponding solution of $L^{Ⓢ2}$, plug this corresponding solution into G_2, and equate to 0.
6. The equation computed above is quadratic in g. Solve the equation for g and choose one solution.

Example

Let $L = n\tau^2 - \tau - (n^2-1)(2n-1)$, $Lu(n) = 0$:

$$d = -(n^2-1)(2n-1)/n$$

$$\begin{aligned} L_T = & n(n+3)(2n+3)(n+1)^2\tau^3 - \\ & n(n+2)\left(2n^3+3n^2-n+1\right)\tau^2 - \\ & (n+2)(n+1)\left(2n^3+3n^2-n+1\right)\tau + \\ & n(n+2)(n-1)(n+1)(2n-1) \end{aligned}$$

$$R = \frac{1}{n}, \quad A = \frac{1}{n}\cdot(g^2 + (3n-2)g + (2n-1)(n-1))$$

$$g = 1-n, \quad \delta = 1-n^2$$

Example (continued)

leading to the output:

$$\hat{L}v(n) = v(n+2) - (2n-1)(n+2)v(n),$$
$$u(n) = \frac{1}{n}v(n) + \frac{1}{n^2-1}v(n+1).$$