# Solving Linear Recurrence Equations

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# An example

Let $u(0) := 0$, $u(1) := 1$, and

$$u(n+2) := \frac{(n+2)((9n+15)u(n) + (6n+28)u(n+1))}{(n+7)(3n+2)}.$$

Then $u(2) = 4$, $u(3) = 12$, $u(4) = 34$, etc.

Can we prove that $u(n)$ is an integer for every positive integer $n$?

# An example

Given only the initial conditions and recurrence relation

$$u(n+2) = \frac{(n+2)((9n+15)u(n)+(6n+28)u(n+1))}{(n+7)(3n+2)}$$

it is difficult to prove that $u(n)$ is an integer sequence.

To prove that $u(n)$ is an integer sequence, it would help to:

- have some combinatorial description (e.g. $u(n)$ is the number of objects with a certain property),
- or, to find a relation between $u(n)$ and some other sequence for which a combinatorial description is known.

# The On-Line Encyclopedia of Integer Sequences

The OEIS has a large number of integer sequences, many of which are given with references, formulas, combinatorial descriptions, and lots of other useful information. Thousands of database sequences satisfy a second order recurrence.

So, if our example is an integer sequence, there is a good chance that it is related to a database sequence.

Database solver:

- For each of the thousands of database recurrences, check if it can be related to our example under gauge transformations ($v(n) = r_0(n)u(n) + r_1(n)u(n+1)$) and term products.
- Since the number of database recurrences is so large, we need fast techniques (a table of $p$-curvatures) to select the right one quickly.

# Back to the example

In our example, the $p$-curvature matches with that of the database sequence named 'A001006.' The term product (multiplying by a hypergeometric term) is trivial, and the gauge transformation is

$$u(n) = r_0(n) \cdot A001006(n) + r_1(n) \cdot A001006(n+1)$$

where

$$r_0(n) = \frac{3(n+1)(n+2)}{(n+4)(n+5)}, \quad r_1(n) = \frac{3(n^2+3n-2)}{(n+4)(n+5)}.$$

# Back to the example

So the implementation of our database solver finds that $u(n)$ equals

$$\frac{3(n+1)(n+2)}{(n+4)(n+5)} \cdot A001006(n) + \frac{3(n^2+3n-2)}{(n+4)(n+5)} \cdot A001006(n+1).$$

This relates our example $u(n)$ to a sequence A001006 (Motzkin numbers) about which many things are known, see the references/comments/formulas in OEIS.

Proving that $u(n)$ is an integer sequence has now become much easier.

(The world wide web URL at the OEIS for A001006 is: http://www.research.att.com/∼njas/sequences/A001006.)

# Switch to Maple worksheets

## Next

- Quickly demonstrate the other solvers included in our program.
- Demonstrate features through the examples.
- Show how the solvers in our program are complementary.
- Software available at http://www.math.fsu.edu/∼glevy.