1. What is a vector space? Short answer: A vector space is a set in which you can add things, and multiply things by scalars.
Longer answer: If V is a vector space, it means the following:
a) V is not empty, in particular, the zero vector is in V.
b) Whenever you pick any two vectors in V, the sum should also be in V.
c) Whenever you pick any vector V, then any scalar multiple of that vector should again be in V.
d) The addition and scalar multiplication satisfy the "usual" rules.

2. What is a subspace of V? Answer: a subspace is some subset of V that is also a vector space. The smallest subspace of V is the set $\{0\}$ which contains only the zero vector, and the largest subspace of V is V itself.

3. Subspaces of $\mathbf{R}^3$ are:
dimension 0, The set that contains only the zero vector.
dimension 1, Any line through the origin.
dimension 2, Any plane through the origin.
dimension 3, $\mathbf{R}^3$ itself.

4. Let $u_1, \ldots, u_n$ be some vectors in $\mathbf{R}^m$. Let $A = (u_1 \cdots u_n)$ be the matrix with those vectors as columns. Then the following things are the same:
Col(A)
SPAN$\{u_1, \ldots, u_n\}$
The set of all linear combinations of $u_1, \ldots, u_n$.
The set of all $c_1u_1 + \cdots c_nu_n$ for all real numbers $c_1, \ldots, c_n$.
The *range* of the linear map $A : \mathbf{R}^n \to \mathbf{R}^m$.
The set $\{Ax \mid x \in \mathbf{R}^n\}$.
The set of all vectors you can get by multiplying A times any vector.
The set $\{b \in \mathbf{R}^m \mid b = Ax \text{ for some vector } x\}$.
The set of all b for which $Ax = b$ is consistent.

5. Given some vector b, how do you check if it is in Col(A)?
Answer: Just check if $Ax = b$ is consistent by row-reducing $(A\ b)$.
Note: If you happen to notice that b is some linear combination of the columns of A (for example, if you notice that b is Column 1 minus 3 times Column 2) then you're done quickly because any linear combination of the columns of A is in Col(A).

6. Given some vector v, how do you check if it is in the Null space of A? Answer: just compute Av, if that's zero then $v \in \text{Nul}(A)$.

7. What exactly is the Column space and the Null space of A?
   If you have a matrix A, we always want to multiply A times some vector. The set of all products you can get that way is $\text{Col}(A)$.
   Now sometimes, when you compute a matrix-vector product Ax, the result could be zero. When that happens, then that vector x is in the Null space of A, so $\text{Nul}(A) = \{x \mid Ax = 0\}$.

   Note that the more $x's$ there are for which $Ax = 0$, the fewer vectors we'll get when we compute Ax for every vector x, so the fewer vectors we'll have in $\text{Col}(A)$. In other words: The bigger $\text{Nul}(A)$ is, the smaller $\text{Col}(A)$ will be. A more precise way to say that is the following: (here dim = dimension)
   $\dim(\text{Nul}(A)) + \dim(\text{Col}(A)) =$ the number of columns of A.

8. Dimension of $V =$ the number of elements of a basis of V.

9. Basis of V. A set $\{u_1, \ldots, u_n\}$ is a basis of V if:
   a) it is a spanning set of V,
   b) and it is linearly independent.

10. Spanning set of V. A set $\{u_1, \ldots, u_n\}$ is a spanning set of V when: $\text{SPAN}\{u_1, \ldots, u_n\} = V$.
    This means that $u_1, \ldots, u_n$ are in V, and if we take all linear combinations of $u_1, \ldots, u_n$ then we get all elements of V.

11. How do you compute a basis of $\text{Nul}(A)$ and of $\text{Col}(A)$?
    In short: every free variable (column without pivot) will give you one basis-element for $\text{Nul}(A)$, whereas every basic variable (column with pivot) will give you one basis-element for $\text{Col}(A)$.
    Remark: This is why the number of basis elements of $\text{Nul}(A)$ plus the number of basis elements of $\text{Col}(A)$ add up to the number of columns of A, which is what is stated in the last equation in item 7 above.

12. More details, how do you get a basis for $\text{Col}(A)$?
    Say $A = (u_1 \cdots u_n)$. We already have a spanning set for $\text{Col}(A)$, namely $\{u_1, \ldots, u_n\}$. What we have to do is check if it is linearly dependent by checking if there are vector(s) among the $u_1, \ldots, u_n$ that are in the SPAN of previous vectors. If there are any, we throw them

away, and what's left is an independent subset of $\{u_1, \ldots, u_n\}$, with the same SPAN, and hence what's left is a basis.
If it is easy to see which one(s) (if any) are in the SPAN of the previous then you can use that, if it's not easy to see then what you do is this: Row-reduce A to row-echelon form or to reduced row-echelon form. For every free variable (= column that doesn't get a pivot), you remove the corresponding vector from $u_1, \ldots, u_n$. So what's left is a subset of $\{u_1, \ldots, u_n\}$ that corresponds to the basic variables (= columns that do get a pivot). That subset will be a basis of the column space of matrix $(u_1 \cdots u_n)$.

13. And how do you get a basis of Nul(A)?
Well, we have to find the solutions of $Ax = 0$. So we row-reduce A, and then write down the general solution of $Ax = 0$. This general solution is a vector and it has zero or more variables in it: the free variables. Pull out these free variables, so you write the general solution as "some free variable" times some vector, plus "another free variable" times another vector, plus .... In other words: you've written the general solution as a linear combination (with weights: the free variables) of some vectors. And those vectors: that's your basis of Nul(A). So you'll have one basis-element for every free variable.

14. If V is some subspace of $\mathbf{R}^n$, then how can I write V as the column space of some matrix A?
So how can I find a matrix A for which Col(A) equals V?
Answer: You have to find a *spanning set* (or better yet: a basis) for V. Then take for matrix A a matrix whose columns are the vectors in your spanning set.

15. If V is some subspace of $\mathbf{R}^m$, then how can I write V as the Null space of some matrix A? So how can I find a matrix A for which $V = \text{Nul}(A)$?
Answer: We have to find *a system of equations* for V, and we want the set of solutions of this system to be V itself. Then for A we simply take the coefficient matrix of that system. Think for a moment about what we have to find, we have to find a system of equations. So we're searching for equations, of which we already know the solutions! (because we want the set of solutions to be V). In Chapter 1 we started with equations and wanted the solutions, but for this problem we want to do the opposite: we're starting with the solution space V

and now we want to find equations.

How are you going to find a system of equations whose solution set is V? First of all, how many equations do you think we'll need? If we had no equations at all, then everything (all of $\mathbf{R}^m$) would be a solution, and the solution set would be $\mathbf{R}^m$, which has dimension m. If we had a system with 1 equation, then the solution set will have dimension $m-1$ (except of course if all coefficients in the equation are 0, lets ignore that case). If we had a system with 2 equations, then the set of solutions will again be smaller (the more equations, the fewer solutions there will be). So with 2 equations, the solution set will have dimension $m-2$ (except of course if the two equations are dependent, if for example the two equations are the same then that doesn't count as two independent equations of course). Looking at it this way, you can see that we'll expect to need this many equations:
m minus the dimension of V.
So if $m = 3$ and V is a plane (dim = 2) we'll need 3–2 = one equation.
And if $m = 3$ and V is a line (dim = 1) we'll need 3–1 = two equations.
Or if $m = 5$ and V is a 3-space (dim = 3) we'll need 5–3 = two equations.

Now how we're actually going to find these equations depends on how V is given to us. If we have a spanning set $u_1, \ldots, u_k$ for V, in other words, if $V = \text{SPAN}\{u_1, \ldots, u_k\}$ then we can do the following: V is the set of all vectors b for which the augmented system $(u_1 \cdots u_k \;\; b)$ is consistent. Now put a variable in every entry of that vector b. Then row-reduce. Then for every zero-row in the coefficient matrix, equate the corresponding right-hand side to zero, and that gives you a linear equation. So for every zero-row on the coefficient side, you get a linear equation from the corresponding right-hand side. Combined, those linear equations form the system that you need.

How many linear equations do we find this way? Matrix $(u_1 \ldots u_k)$ has m rows (because $u_1, \ldots, u_k$ were in $\mathbf{R}^m$) and the rank of this matrix is the dimension of $\text{SPAN}\{u_1, \ldots, u_k\} = V$. The number of zero-rows = number of rows minus rank = $m - \dim(V)$. So that's how many equations we'll get for V.

16. Coordinate vectors. Let B be a basis of a vector space V and let v be some element of v. How to find $[v]_B$?
Answer: Write v as a linear combination of the vectors in B, and put

the weights in this linear combination in a vector, the result is $[v]_B$. So this $[v]_B$ will have one entry (the weight in that linear combination) for each element of B. The number of entries in your $[v]_B$ must equal the number of vectors in the basis B!

If v and the elements of B are vectors, then you can compute as follows: Take matrix $(B\ v)$ and row-reduce. If this system is inconsistent, then your vector v is NOT in SPAN(B). If the system is consistent then v is in SPAN(B).

If B is a basis of V then SPAN$(B) = V$ and if v is in V then v must be in SPAN(B) so in that case the system $(B\ v)$ must be consistent. If we compute the reduced row echelon form then we can easily read off the solution of system $(B\ v)$ and hence the coordinate vector $[v]_B$.

If you have to compute the coordinate vector of a whole bunch of vectors, say we have to compute $[v_1]_B$, $[v_2]_B$ and $[v_3]_B$, then row-reduce the augmented matrix $(B\ v_1 v_2 v_3)$. This computation will show exactly which of those vectors v_1, v_2, v_3 are actually in SPAN(B) and for those, you can immediately read off the coordinate vector from the reduced row echelon form of $(B\ v_1 v_2 v_3)$.

17. Coordinate vectors, examples.
Example 1. Let V be the set of all polynomials in t of degree ≤ 2. Let $B = 1, t, t^2$ and let $v = 8t + 5 - 3t^2$. What is $[v]_B$?
Well, v equals: 5 times the first element of B plus 8 times the second plus -3 times the third element of B. Hence:
$[v]_B = \begin{pmatrix} 5 \\ 8 \\ -3 \end{pmatrix}$.

Example 2. Let $B = \cos(t), \sin(t)$ and let $v = \cos(\pi/6 + t)$. What is $[v]_B$? Well, according to trigonometry, $\cos(a + b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and so $v = \cos(\pi/6+t) = \cos(\pi/6)\cos(t) - \sin(\pi/6)\sin(t) = \frac{1}{2}\sqrt{3}\cos(t) - \frac{1}{2}\sin(t)$. So we need $\frac{1}{2}\sqrt{3}$ times the first element of B plus $-1/2$ times the second, and hence:
$[v]_B = \begin{pmatrix} \frac{1}{2}\sqrt{3} \\ -\frac{1}{2} \end{pmatrix}$.

Example 3. $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 4 & 2 \\ 0 & 1 \end{pmatrix}$.

Compute $[v]_B$. Answer: this vector $[v]_B$ will have 3 entries because B contains 3 "vectors" (which look like matrices). If the entries of $[v]_B$ are c_1, c_2, c_3 then we have to solve this system: $v =$
c_1 times first element of B, plus c_2 times second, plus c_3 times third. So we have to find real numbers c_1, c_2, c_3 for which that holds. Lets spell it out:

$$\begin{pmatrix} 4 & 2 \\ 0 & 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + c_3 \begin{pmatrix} 3 & 2 \\ 1 & 1 \end{pmatrix}.$$

Of course, where exactly we put those numbers $4, 2, 0, 1$ does not matter much, whether we put them in a square (in a matrix) or on a line (in a vector) that should make no difference to c_1, c_2, c_3. So we can rewrite the system as:

$$\begin{pmatrix} 4 \\ 2 \\ 0 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} + c_3 \begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix}.$$

The augmented matrix for this system is this, which row-reduces to:

$$\begin{pmatrix} 1 & 1 & 3 & 4 \\ 1 & 2 & 2 & 2 \\ 1 & 3 & 1 & 0 \\ 1 & 4 & 1 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

and so $c_1 = -6$, $c_2 = 1$, and $c_3 = 3$ so the answer is $[v]_B = \begin{pmatrix} -6 \\ 1 \\ 3 \end{pmatrix}$.

Note: we see that $[v]_B$ is the top part of the right-hand side column in the reduced row echelon form. When you give $[v]_B$ you must not write down the entire column from that matrix because that would have too many entries, remember that the number of entries of $[v]_B$ must always equal the number of things in B, which is 3 in this example.

Example 4. If $B = u_1,\ 5u_2 - u_1,\ 13u_2 - u_3$ and $[v]_B = \begin{pmatrix} 3 \\ -7 \\ \alpha \end{pmatrix}$ then what is v? Answer: $3u_1 - 7\,(5u_2 - u_1) + \alpha\,(13u_2 - u_3)$ (whatever these expressions may be doesn't matter, all you have to know here is that if $[v]_B$ has entries $3, -7, \alpha$ then v must be 3 times first element of B, plus -7 times second, plus α times third).