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`http://www.math.fsu.edu/~hoeij/spring2004/`

1. What is the *vector projection* of u on w?
   Answer: It is the scalar multiple of w that is as close as possible to u.

   So what does this mean? Well, draw the line through w and the origin, lets call that line W. If u is on that line, then the vector projection of u on w is u itself. If u is not on that line, then pick the point on that line that is as close as possible to u, and then that point is the vector projection of u on w.

2. How do you compute the vector projection of vector u on vector w?
   Answer: Compute these two numbers: $u \cdot w$ and $w \cdot w$. Then take the quotient. Multiply that by w and you get the vector projection of u on w:

   $$\text{proj}(u \text{ on } w) \quad = \quad \frac{u \cdot w}{w \cdot w} w$$

   Since this is a scalar (the quotient of those two dot-products) times w, we see that the projection of u on w is always on the line $W = \text{SPAN}(w)$

3. Let W be some subspace of $\mathbf{R}^n$ and let u be some element of $\mathbf{R}^n$. What is the *vector projection* of u on W?
   Answer: It is the element of W that is as close as possible to u. So if u is in W then the projection of u on W is just u itself. If u is not in W, then pick the point in W that is the closest to u, and then that point is the vector projection of u on W.

4. How do you compute the vector projection of vector u on W?
   Answer: First you need an *orthogonal basis* of W. Suppose that $w_1, \ldots, w_k$ is an orthogonal basis of W (how to find an orthogonal basis of W is the subject of items 18,19). Then

   $$\text{proj}(u \text{ on } W) = \text{proj}(u \text{ on } w_1) + \text{proj}(u \text{ on } w_2) + \cdots + \text{proj}(u \text{ on } w_k)$$

   in other words, the projection of u on W is

   $$\text{proj}(u \text{ on } W) \quad = \quad \frac{u \cdot w_1}{w_1 \cdot w_1} w_1 \quad + \quad \frac{u \cdot w_2}{w_2 \cdot w_2} w_2 \quad + \cdots + \quad \frac{u \cdot w_k}{w_k \cdot w_k} w_k$$

   This only works if $w_1, \ldots, w_k$ is an orthogonal basis of W.

5. What does $u \perp v$ mean?
   Answer: $u \perp v$ means that u is orthogonal to v, which in turn means that the dot-product (the inner product) of u and v is zero, so $u \cdot v = 0$.
   This happens when $u = 0$, or when $v = 0$, or when u, v are perpendicular (the angle between them is 90^o).

6. If W is some subspace of $\mathbf{R}^n$ then what does $W^\perp$ mean?
   Answer: It's the set of those vectors in $\mathbf{R}^n$ that are orthogonal to *every* element of W.

7. If u is some vector of $\mathbf{R}^n$ and if W is some subspace of $\mathbf{R}^n$ then we can write
   $$u \;=\; \text{proj}(u \text{ on } W) \quad + \quad (\; u - \text{proj}(u \text{ on } W)\;)$$
   Now proj(u on W) is in W, while $u -$ proj(u on W) is in $W^\perp$. So we just wrote:
   $$u \;=\; \text{something in } W \;\; + \quad \text{something in } W^\perp$$
   Since vector u was an arbitrary vector, we see that every vector of $\mathbf{R}^n$ can be written as the sum of something in W plus something in $W^\perp$. A mathematician would formulate this fact in a very compact way, namely as follows: $\mathbf{R}^n = W + W^\perp$

8. If I know some basis (or a spanning set) $w_1, \ldots, w_k$ of W, then how do I get a basis of $W^\perp$?
   Answer: Since $w_1, \ldots, w_k$ is a basis (or a spanning set) of W, it means that W is the column space of matrix $A = (w_1 \; w_2 \; \cdots \; w_k)$. Now take the ***transpose*** of that matrix, and then take the *Nullspace* of that. So $W^\perp = \text{Nul}(A^T)$.

9. An example, let e_1, e_2, e_3, e_4, e_5 be the standard basis of $\mathbf{R}^5$ and suppose that e_3, e_4 is a basis of W. Give a basis of $W^\perp$.
   Answer: The general method is to take $A = (e_3 \; e_4)$, then take the transpose, and then the Nullspace, and a basis of that. Then we'll find the basis e_1, e_2, e_5 of $W^\perp$ (verify this example for yourself!). This example is easy because the matrix A^T is already in reduced row echelon form so we don't have to row-reduce in this example.
   This example is also easy for another reason, you see, for some vector u to be in $W^\perp$ it must be orthogonal to all elements in the basis of W. So u must be orthogonal to e_3, so $e_3 \cdot u = 0$, but that simply means that the third entry of u is zero. And u must be orthogonal to e_4 but that just means the 4'th entry is zero. So u can be any vector whose 3'rd and 4'th entries are zero, and it is clear that e_1, e_2, e_5 is a basis of such vectors, so e_1, e_2, e_5 must be a basis of $W^\perp$.

10. If W is a subspace of $\mathbf{R}^n$ then $n = \dim(W) + \dim(W^\perp)$.
    So the dimension of $W^\perp$ is n minus $\dim(W)$. If we take $n = 3$ then this means that if W is a line then $W^\perp$ is a plane and if W is a plane then $W^\perp$ is a line.

11. Suppose I don't have a basis (or spanning set) of W, but instead, I have a system of equations for W. Say that the coefficient matrix of that system is A, in other words, say that W is the Nullspace of A. Now how do I find

$W^\perp$ in this situation?
Answer: We could of course compute a basis of W (because we know how to compute a basis of Nul(A)) and then do as in item 8, but there is an easier way. Let $R_1, R_2, \ldots, R_l$ be the rows of A. Now if $w \in W$ then $Aw = 0$ so then $R_1 w = 0$, $R_2 w = 0$, etc. Now let $v_1, v_2, \ldots, v_l$ be the transposes of $R_1, \ldots, R_l$. Then $R_1 w$ is just $v_1 \cdot w$. So we get $v_1 \cdot w = 0$, $v_2 \cdot w = 0$, etc., and this is true for every $w \in W$. Therefore, $v_1 \in W^\perp$, and $v_2 \in W^\perp$, etc.
Conclusion is the following: If W is the Nullspace of some matrix A, then the ***transposes of the rows*** of A will form a ***spanning set for*** $W^\perp$ (they'll form a basis if they're independent).
One-line summary: If $W = \text{Nul}(A)$ then $W^\perp = \text{RowSpace}(A)$.

12. As an example, take $W = \{\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x + 2y + 3z = 0\}$. This is a 2-dimensional subspace of $\mathbf{R}^3$ (a plane in $\mathbf{R}^3$). This W is given by a system of linear equations (just 1 equation but that's OK). The matrix of that system is (1 2 3). Following item 11 we see that $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ is a basis of $W^\perp$.

13. Another example, what if V is a vector space with basis $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ then how to get a basis of $V^\perp$?
Answer: put the basis vector(s) in a matrix, then take the transpose (that's (1 2 3) in this example) and then the Nullspace. We have a pivot in Column 1, so x_1 is basic, and x_2, x_3 are free. We get $x_1 = -2x_2 - 3x_3$, $x_2 = x_2$, $x_3 = x_3$ so the solutions of matrix (1 2 3) are $\begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix}$ and writing this as x_2 times a vector plus x_3 times a vector we get the following two vectors $\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$. These two vectors form a basis of the Nullspace of (1 2 3), and hence a basis of $V^\perp$.
Note that the vector space V in this example is the same as the vector space $W^\perp$ from the previous example. But then $V^\perp$ must be $(W^\perp)^\perp$ which is the same as W. So our basis for $V^\perp$ is also a basis for the space W from the previous example.

14. If W is a subspace of $\mathbf{R}^n$ then $(W^\perp)^\perp = W$.

15. What's an orthogonal set?
    Answer: It's a set where every element is orthogonal to every other element.
    How do I check if $\{w_1, w_2, \ldots, w_k\}$ is an orthogonal set?
    Answer: You check that each of them is orthogonal to all the previous ones, so you check that $w_2 \cdot w_1 = 0$, then check that $w_3 \cdot w_1 = 0$ and $w_3 \cdot w_2 = 0$, then check that $w_4 \cdot w_1 = 0$, $w_4 \cdot w_2 = 0$, $w_4 \cdot w_3 = 0$, etc.

16. What's an orthogonal basis of a vector space W?
    Answer: a basis where every element is orthogonal to every other element.

17. If $w_1, \ldots, w_k$ are some vectors, what's the quickest way to see if they form an orthogonal basis of W?
    Answer: First of all, they must all be in W. Second, the zero-vector must not be among $w_1, \ldots, w_k$. Furthermore, k, the number of vectors in your set, must be equal to the dimension of V. Finally, check that they form an orthogonal set (see item 15).

    Don't I have to check that $w_1, \ldots, w_k$ are linearly independent to make sure that I have a basis of W?
    Answer: *an orthogonal set without zero-vectors* is automatically linearly independent.

18. How do I get an *orthogonal basis* of W?
    Answer: first, you need a basis (or a spanning set, that's OK too) for W. Say that $u_1, \ldots, u_k$ is a spanning set of W. Now you follow the following process, called the Gram-Schmidt process:

    Take $v_1 = u_1$.
    Take v_2 to be u_2 MINUS the vector projection of u_2 on all previous v's.
    Take v_3 to be u_3 MINUS the vector projection of u_3 on all previous v's.
    Take v_4 to be u_4 MINUS the vector projection of u_4 on all previous v's.
    etc.
    If any of these v's are zero, then just throw that one away (this only happens if the u's were linearly dependent).
    The remaining v's (the non-zero v's) will be an orthogonal basis of W.

19. Can you spell that out in some more detail, how to get an orthogonal basis of W if I have some spanning set $u_1, \ldots, u_k$ of W?
    Answer: Follow the previous item, and just plug in the these vector projections. So you get:
    $v_1 = u_1$
    $v_2 = u_2 - \text{proj}(u_2 \text{ on } v_1)$
    $v_3 = u_3 - \text{proj}(u_3 \text{ on SPAN}\{v_1, v_2\})$
    $v_4 = u_4 - \text{proj}(u_4 \text{ on SPAN}\{v_1, v_2, v_3\})$, etc.
    If we spell this out with the formula for the vector projection (see items 2

and 4) then we get:

$v_1 = u_1$

$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1$

$v_3 = u_3 - \left(\frac{u_3 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{u_3 \cdot v_2}{v_2 \cdot v_2} v_2 \right)$

$v_4 = u_4 - \left(\frac{u_4 \cdot v_1}{v_1 \cdot v_1} v_1 + \frac{u_4 \cdot v_2}{v_2 \cdot v_2} v_2 + \frac{u_4 \cdot v_3}{v_3 \cdot v_3} v_3 \right)$, etc.

In step 3, make sure that you use u_3 and the previous v's (not the previous u's). In step 4, use u_4 and the previous v's (not the previous u's).

20. Example, let $u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, $u_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $u_3 = \begin{pmatrix} 0 \\ 1 \\ 4 \\ 9 \\ 16 \end{pmatrix}$ and $u = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 3 \end{pmatrix}$.

Let $W = \text{SPAN}(u_1, u_2, u_3)$. Find the vector projection of u on W, i.e. find the vector in W that is as close as possible to u.

Answer: if u_1, u_2, u_3 were an orthogonal set, we could use the formula in item 4 (the w's in item 4 would then be the u's here). But, u_1, u_2, u_3 are not orthogonal, for example $u_1 \cdot u_2 \neq 0$. We'll have to fix that with Gram-Schmidt. We take:

$v_1 = u_1$

$v_2 = u_2 - \frac{0\cdot1+1\cdot1+2\cdot1+3\cdot1+4\cdot1}{1^2+1^2+1^2+1^2+1^2} u_1 = \begin{pmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{pmatrix}$

$v_3 = u_3 - \left(\frac{0\cdot1+1\cdot1+4\cdot1+9\cdot1+16\cdot1}{1^2+1^2+1^2+1^2+1^2} u_1 + \frac{(-2)\cdot0+(-1)\cdot1+0\cdot4+1\cdot9+2\cdot16}{(-2)^2+(-1)^2+0^2+1^2+2^2} u_2 \right) = \begin{pmatrix} 2 \\ -1 \\ -2 \\ -1 \\ 2 \end{pmatrix}$

Now that we have an *orthogonal basis* v_1, v_2, v_3 of the vector space W, we are ready to compute the vector projection of u on W with the formula from item 4 (the w's in item 4 are the v's here).

$\text{proj}(u \text{ on } W) = \frac{5}{5} v_1 + \frac{5}{10} v_2 + \frac{7}{14} v_3$. If we compute that, we get u itself (this means that u was actually in W, so the vector in W closest to u is then of course u itself). Let's compute $\text{proj}(u \text{ on } W)$ for another u, say

$u = \begin{pmatrix} -2 \\ 0 \\ 3 \\ 2 \\ 2 \end{pmatrix}$. Then $\text{proj}(u \text{ on } W) = \frac{5}{5} v_1 + \frac{10}{10} v_2 + \frac{-8}{14} v_3 = \begin{pmatrix} -15/7 \\ 4/7 \\ 15/7 \\ 18/7 \\ 13/7 \end{pmatrix}$.

Application: if $f(x)$ is a function that takes values $-2, 0, 3, 2, 2$ (the entries of u) at $x = 0, 1, 2, 3, 4$ then the quadratic function that best approximates this takes as values "the entries of $\text{proj}(u \text{ on } W)$" at $x = 0, 1, 2, 3, 4$.