Linear algebra, Answers test 3.

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Write down your name and SSN.

1. Let

$$u = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

(a) (10 points). What is $\cos(\theta)$ where θ is the angle between u and v? What is the length of u and v?

$||u|| = \sqrt{1^2 + 1^2 + 2^2 + 2^2} = \sqrt{10}$
$||v|| = \sqrt{1^2 + 2^2 + 1^2 + 2^2} = \sqrt{10}$
$\cos(\theta) = \frac{u \cdot v}{||u||\ ||v||} = (1 \cdot 1 + 1 \cdot 2 + 2 \cdot 1 + 2 \cdot 2)/(\sqrt{10} \cdot \sqrt{10}) = \frac{9}{10}$.

(b) (10 points). What is the vector projection of u onto v, and what is the length of this projection?

$$\frac{u \cdot v}{v \cdot v} v = \frac{9}{10} v = \begin{pmatrix} 0.9 \\ 1.8 \\ 0.9 \\ 1.8 \end{pmatrix}.$$

The length of that is nine tenth of the lenght of v, so that is $\frac{9}{10}\sqrt{10}$.

(c) (15 points). Let $A = (u \ \ v)$. Compute a matrix B such that the null space of B is the column space of A

$$\mathcal{CS}(A) = \mathcal{NS}(B).$$

Let $K = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$ and check when the system $(A|K)$ is consistent.
After row-reducing this matrix you get

$$\begin{pmatrix} 1 & 0 & 2\,k_1 - k_2 \\ 0 & 1 & -k_1 + k_2 \\ 0 & 0 & -3k_1 + k_2 + k_3 \\ 0 & 0 & -2k_1 + k_4 \end{pmatrix}$$

So K is in the column space when this system is consistent, which is when $-3k_1 + k_2 + k_3 = 0$ and when $-2k_1 + k_4 = 0$. Now $-3k_1 + k_2 + k_3 = (-3\ 1\ 1\ 0)K$ and $-2k_1 + k_4 = (-2\ 0\ 0\ 1)$ so these two equations hold precisely when

$$\begin{pmatrix} -3 & 1 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix} K = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

in other words when K is in the null space of matrix

$$B = \begin{pmatrix} -3 & 1 & 1 & 0 \\ -2 & 0 & 0 & 1 \end{pmatrix}.$$

So the null space of this B is the column space of A. Note that there can also be other correct answers. If your answer is different from this B but is row-equivalent to this B then it has the same null space as this B and so then it is correct as well.

(d) (10 points). For which values of x and y is the vector

$$w = \begin{pmatrix} x+1 \\ x \\ y \\ -y \end{pmatrix}$$

an element of SPAN($\{u, v\}$)?

There are two ways to do this. One way is to row-reduce $(uv|w) = (A|w)$. On the left of the seperator | you get two zero-rows after row-reduction. So the corresponding entries on the right must be zero, and this way you get two linear equations in x and y. If you solve those you find the answer.
Another way to answer this question is the following: w is in SPAN($\{u, v\}$) is the column space of A, but we have seen that that is the same as the null space of matrix B. So SPAN($\{u, v\}$) $= \mathcal{CS}(A) = \mathcal{NS}(B)$,

so $w \in \text{SPAN}(\{u, v\})$ precisely when $w \in \mathcal{NS}(B)$, which is true precisely when $Bw = 0$. So calculate Bw and you get

$$Bw = \begin{pmatrix} -3(x+1) + x + y \\ -2(x+1) - y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

If you solve these two equations you get $x = -5/4$, $y = 1/2$.

2. (25 points) Consider the following vectors

$$u = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Is u as a linear combination of v_1, v_2, v_3? If so, give that linear combination and verify that your answer is correct.

We have to row-reduce the matrix $(v_1\ v_2\ v_3|u)$. The result is

$$\begin{pmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

Therefore we see that u is a linear combination of v_1, v_2, v_3, namely $u = -2v_1 + v_2 + 2v_3$.

3. Let

$$A = \begin{pmatrix} 2 & 2 & -2 \\ -1 & 1 & 1 \\ 0 & 2 & 0 \end{pmatrix}.$$

(a) (10 points) Compute the characteristic polynomial of A.

We have to calculate the determinant of

$$\lambda I - A = \begin{pmatrix} -2+\lambda & -2 & 2 \\ 1 & -1+\lambda & -1 \\ 0 & -2 & \lambda \end{pmatrix}$$

and the result is $\lambda^3 - 3\lambda^2 + 2\lambda$. Calculation errors can often happen. That is why you should always check the result in the following way: the coefficient of λ^{n-1} ($n = 3$ in this exercise) should be equal to -1 times the trace of the matrix, the sum of the diagonal. The sum of the diagonal equals $2+1+0=3$, and indeed the coefficient of λ^2 is -3. If you get something different then -3 then finding the error in your computation can often take more time than doing the whole computation all over again, so it is often better to start all over again. A second check you can do is that the constant term of the characteristic polynomial should be equal to $(-1)^n$ times the determinant. If we row-reduce A we quickly find a zero row so the determinant equals zero, so the constant term should be zero. Note that since the determinant (which is the product of the eigenvalues) is zero it follows that one of the eigenvalues must be zero.

(b) (5 points) Compute the eigenvalues of A. Use the trace and the determinant of A as a way to check the correctness your answer.

$\lambda^3 - 3\lambda^2 + 2\lambda = \lambda(\lambda-1)(\lambda-2)$ so the eigenvalues are $0, 1, 2$. The sum is 3 which equals the trace, the product equals zero which is the determinant. Note that it is a good idea to always do this check. Why? Well, suppose that in the next question you row-reduce that matrix and you find no non-zero element of the null space, so you don't find a non-zero eigenvector. That's impossible, there should always be a non-zero eigenvector for each eigenvalue. So in such a situation you know that a computation error has been made: either the row-reduction is wrong, or the eigenvalue is wrong. But if you did these checks then you know almost certainly that the eigenvalues are right, so at least you know at that point that the error was in the computation of the null space and not somewhere else.

(c) (15 points) For each eigenvalue, compute one corresponding non-zero eigenvector.

For $\lambda = 0$, $\lambda = 1$ and $\lambda = 2$ we have to calculate one non-zero element of the null space of $\lambda I - A$. So for each of these values for λ, compute that matrix $\lambda I - A$ and row-reduce it. You should get at least one zero-row after the row-reduction because there must be a non-zero element of the null-space.

$\lambda = 0$. The rref of $0I - A$ is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

the null space of that is

$$\{\begin{pmatrix} x_3 \\ 0 \\ x_3 \end{pmatrix} | x_3 \in \mathbb{R}\}$$

but we only need to give one non-zero eigenvector so we can substitute any non-zero real number for x_3. Because we want to use these vectors in the next question, it's best to substitute a real number that gives the simplest looking result. Take for example $x_3 = 1$ and we get

$$u_0 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

It is easy to check that $Au_0 = 0u_0 = 0$.

$\lambda = 1$. The rref of $1I - A$ is

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

the null space of that is

$$\{\begin{pmatrix} x_3 \\ \frac{1}{2}x_3 \\ x_3 \end{pmatrix} | x_3 \in \mathbb{R}\}$$

but we only need to give one non-zero eigenvector so we can substitute any non-zero real number for x_3. Take for example $x_3 = 2$ so we get a vector without fractions and we get

$$u_1 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

It is easy to check that $Au_1 = 1u_1$.

$\lambda = 2$. The rref of $2I - A$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

the null space of that is

$$\{\begin{pmatrix} 0 \\ x_3 \\ x_3 \end{pmatrix} | x_3 \in \mathbb{R}\}$$

but we only need to give one non-zero eigenvector so we can substitute any non-zero real number for x_3. Take for example $x_3 = 1$ so we get and we get

$$u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

It is easy to check that $Au_2 = 2u_2$.

(d) (10 bonus points). Write the vector

$$w = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

as a linear combination of these three eigenvectors. Use that linear combination to calculate $B \cdot w$ where $B = A^{10}$ without doing a lot of matrix multiplications.

Row-reduce $(u_0 u_1 u_2 | w)$ to

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

so we see that $w = -u_0 + u_1 - u_2$.

Now

$$Bw = -Bu_0 + Bu_1 - Bu_2 = -A^{10}u_0 + A^{10}u_1 - A^{10}u_2$$

Now A times an eigenvector u_i with eigenvalue λ_i equals $\lambda_i u_i$ so A^{10} times an eigenvector u_i with eigenvalue λ_i equals $\lambda_i^{10} u_i$. So $A^{10}u_0 = 0^{10}u_0 = 0$, $A^{10}u_1 = 1^{10}u_1 = u_1$ and $A^{10}u_2 = 2^{10}u_2 = 1024u_2$. So

$$Bw = u_1 - 1024u_2 = \begin{pmatrix} 2 \\ -1023 \\ -1022 \end{pmatrix}.$$