Linear algebra, test 4

Write down your name and SSN. The number of points add up to 105, so there are 5 bonus points.

1.

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

Let $V = \mathrm{SPAN}(\{u_1, u_2, u_3\})$.

(a) (4 points). Compute a matrix B such that the Null space of B equals V.

(b) (2 points). Compute a non-zero vector that is orthogonal to every vector in V.

(c) (2 points). What is the dimension of V?

(d) (2 points). Compute: the projection of u_2 on u_1, and the projection of u_3 on u_1.

(e) (8 points). Give an orthonormal basis B of V.

(f) (6 points). Compute the coordinate vectors of u_1, u_2, u_3 with respect to B; $[u_1]_B$, $[u_2]_B$, $[u_3]_B$.

(g) (4 points). Which of the following are in V (hint: use matrix B).

$$w_1 = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, w_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 2 \end{pmatrix}.$$

2. Let

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{pmatrix}$$

Compute the following:

(a) (4 points). The reduced row echelon form of A.

(b) (2 points). The rank of A.

(c) (2 points). A basis for the column space $\mathcal{CS}(A)$ of A.

(d) (4 points). An orthonormal basis for the column space.

(e) (2 points). A basis for the null space $\mathcal{NS}(A)$ of A.

(f) (2 points). A basis for the row space $\mathcal{RS}(A)$ of A.

(g) (3 points). Are the columns of A linearly dependent? If so, then give a linear relation.

(h) (3 points). Are the rows of A linearly dependent? If so, then give a linear relation.

3. Let

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_6 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) (4 points). Compute the reduced row echelon form of $B = (u_1\ u_2\ u_3\ u_4\ u_5\ u_6)$.

(b) (10 points). Compute a basis of the null space, a basis of the row space and a basis of the column space of B.

(c) (6 points). Give a basis for each of the following vector spaces: $\mathrm{SPAN}(\{u_1\})$, $\mathrm{SPAN}(\{u_1, u_2\})$, $\mathrm{SPAN}(\{u_1, u_2, u_3\})$, $\mathrm{SPAN}(\{u_1, u_2, u_3, u_4\})$, $\mathrm{SPAN}(\{u_1, u_2, u_3, u_4, u_5\})$, $\mathrm{SPAN}(\{u_1, u_2, u_3, u_4, u_5, u_6\})$.

(d) (5 points). Whenever $\mathrm{SPAN}(\{u_1, u_2, \ldots, u_n\}) = \mathrm{SPAN}(\{u_1, u_2, \ldots, u_n, u_{n+1}\})$ express u_{n+1} as a linear combination of $u_1, u_2, \ldots, u_n$.

4. Let $u_1 = x^3 - 4x$, $u_2 = x^2 - x - 2$, $u_3 = x^3 - x^2 - x - 2$, $u_4 = x^2 - 4$.

Let $v_1 = x - 2$, $v_2 = x^2 - 2x$, $v_3 = x^3 - 2x^2$.

Let $V = \mathrm{SPAN}(\{u_1, u_2, u_3, u_4\})$.

(a) (6 points) V is a subspace of P_3 and the polynomials u_1, u_2, u_3, u_4, v_1, v_2, v_3 are elements of P_3. Give the coordinate vectors of these polynomials with respect to the basis $\{1, x, x^2, x^3\}$ of P_3.

(b) (10 points). Show that $B = \{v_1, v_2, v_3\}$ is a basis for V and compute $[u_1]_B$, $[u_2]_B$, $[u_3]_B$ and $[u_4]_B$.

(c) (3 points). Are u_1, u_2, u_3, u_4 linearly dependent or independent?

(d) (3 points). Is $\{f \in P_3 | f(2) = 0\}$ a subspace of P_3? If so, can you find a basis for it?

5. (8 points). Let A be a 5 by 7 matrix for which the rank is 4. Compute the following:

- The dimension of the null space of A.
- The dimension of the column space of A.
- The dimension of the row space of A.
- The dimension of the null space of A^T.
- The dimension of the column space of A^T.
- The dimension of the row space of A^T.
- Are the rows of A linearly dependent or independent?
- Are the columns of A linearly dependent or independent?

Good luck!