Linear algebra, test 4

Write down your name and SSN. The number of points adds up to 105, so there are 5 bonus points.

1.

\[ u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \end{pmatrix}. \]

Let \( V = \text{SPAN}\{u_1, u_2, u_3\} \).

(a) (4 points). Compute a matrix \( B \) such that the Null space of \( B \) equals \( V \).

(b) (2 points). Compute a non-zero vector that is orthogonal to every vector in \( V \).

(c) (2 points). What is the dimension of \( V \)?

(d) (2 points). Compute the projection of \( u_2 \) on \( u_1 \), and the projection of \( u_3 \) on \( u_1 \).

(e) (8 points). Give an orthonormal basis \( B \) of \( V \).

(f) (6 points). Compute the coordinate vectors of \( u_1, u_2, u_3 \) with respect to \( B \); \([u_1]_B, [u_2]_B, [u_3]_B\).

(g) (4 points). Which of the following are in \( V \) (hint: use matrix \( B \)).

\[ w_1 = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 2 \end{pmatrix}. \]

2. Let

\[ A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{pmatrix} \]

Compute the following:
(a) (4 points). The reduced row echelon form of $A$.
(b) (2 points). The rank of $A$.
(c) (2 points). A basis for the column space $\text{CS}(A)$ of $A$.
(d) (4 points). An orthonormal basis for the column space.
(e) (2 points). A basis for the null space $\text{NS}(A)$ of $A$.
(f) (2 points). A basis for the row space $\text{RS}(A)$ of $A$.
(g) (3 points). Are the columns of $A$ linearly dependent? If so, then give a linear relation.
(h) (3 points). Are the rows of $A$ linearly dependent? If so, then give a linear relation.

3. Let

$$
\begin{align*}
  u_1 &= \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \\
  u_2 &= \begin{pmatrix} 1 \\ -1 \\ 1 \\ 1 \end{pmatrix}, \\
  u_3 &= \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \\
  u_4 &= \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \\
  u_5 &= \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \\
  u_6 &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\end{align*}
$$

(a) (4 points). Compute the reduced row echelon form of $B = (u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6)$.
(b) (10 points). Compute a basis of the null space, a basis of the row space and a basis of the column space of $B$.
(c) (6 points). Give a basis for each of the following vector spaces:
   - $\text{SPAN}\{u_1\}$
   - $\text{SPAN}\{u_1, u_2\}$
   - $\text{SPAN}\{u_1, u_2, u_3\}$
   - $\text{SPAN}\{u_1, u_2, u_3, u_4\}$
   - $\text{SPAN}\{u_1, u_2, u_3, u_4, u_5\}$
   - $\text{SPAN}\{u_1, u_2, u_3, u_4, u_5, u_6\}$
(d) (5 points). Whenever $\text{SPAN}\{u_1, u_2, \ldots, u_n\} = \text{SPAN}\{u_1, u_2, \ldots, u_n, u_{n+1}\}$ express $u_{n+1}$ as a linear combination of $u_1, u_2, \ldots, u_n$.

4. Let $u_1 = x^3 - 4x$, $u_2 = x^2 - x - 2$, $u_3 = x^3 - x^2 - x - 2$, $u_4 = x^2 - 4$.
Let $v_1 = x - 2$, $v_2 = x^2 - 2x$, $v_3 = x^3 - 2x^2$.

Let $V = \text{SPAN}\{u_1, u_2, u_3, u_4\}$.

(a) (6 points) $V$ is a subspace of $P_3$ and the polynomials $u_1, u_2, u_3, u_4, v_1, v_2, v_3$ are elements of $P_3$. Give the coordinate vectors of these polynomials with respect to the basis $\{1, x, x^2, x^3\}$ of $P_3$.
(b) (10 points). Show that $B = \{v_1, v_2, v_3\}$ is a basis for $V$ and compute $[u_1]_B$, $[u_2]_B$, $[u_3]_B$ and $[u_4]_B$.
(c) (3 points). Are $u_1, u_2, u_3, u_4$ linearly dependent or independent?
(d) (3 points). Is $\{f \in P_3 | f(2) = 0\}$ a subspace of $P_3$? If so, can you find a basis for it?

5. (8 points). Let $A$ be a 5 by 7 matrix for which the rank is 4. Compute the following:
• The dimension of the null space of $A$.
• The dimension of the column space of $A$.
• The dimension of the row space of $A$.
• The dimension of the null space of $A^T$.
• The dimension of the column space of $A^T$.
• The dimension of the row space of $A^T$.
• Are the rows of $A$ linearly dependent or independent?
• Are the columns of $A$ linearly dependent or independent?

*Good luck!*