Linear algebra, answers test 4.

Write down your name and SSN. The number of points add up to 105, so there are 5 bonus points.

1.

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \quad u_2 = \begin{pmatrix} -3 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 5 \end{pmatrix}.$$

Let $V = \text{SPAN}(\{u_1, u_2, u_3\})$.

(a) (4 points). Compute a matrix B such that the Null space of B equals V.

Answer: Let $k = (k_1\ k_2\ k_3\ k_4)^T$ and row-reduce $(u_1 u_2 u_3 | k)$ to r.e.f. (you don't need r.r.e.f). Result

$$\left(\begin{array}{ccc|c} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 2k_1 + 5k_2 - k_4 \end{array}\right).$$

So for K to be in V we need $2k_1 + 5k_2 - k_4$ to be zero. The coefficient matrix of this equation is $B = (2\ 5\ 0\ -1)$, so when $BK = 0$ then $K \in V$. So the null space of B is V.

(b) (2 points). Compute a non-zero vector that is orthogonal to every vector in V.

Answer:

$$B^T = \begin{pmatrix} 2 \\ 5 \\ 0 \\ -1 \end{pmatrix}$$

Why? Well, if $v \in V$ then the matrix product Bv equals the dot product $B^T\ v$. Now Bv is zero because v is in V which is the null space of B. So the column vector B^T has dot product zero with every $v \in V$.

(c) (2 points). What is the dimension of V?

Answer: From the r.e.f. in 1a) we see that u_1, u_2, u_3 are linearly independent so the span of that has dimension 3. This is always the way to compute the dimension of the span of a bunch of vectors, row-reduce and count the number of row-leaders (same as the number of non-zero rows).

Another way one could have found the answer: V is the null space of B, and matrix B has 4 columns and 1 row-leader, so 1 basic variable and 3 free variables, so the null space of B has dimension 3.

(d) (2 points). Compute: the projection of u_2 on u_1, and the projection of u_3 on u_1.

Answer:

$$\begin{pmatrix} -1 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 2 \\ 0 \\ 0 \\ 4 \end{pmatrix}$$

(e) (8 points). Give an orthonormal basis B of V.

Answer: Gram-Schmidt gives the following result (note that of the three vector projections we need to calculate already two are given in 1d).

$$v_1 = u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}$$

$$v_2 = u_2 - \begin{pmatrix} -1 \\ 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v_3 = u_3 - \begin{pmatrix} 2 \\ 0 \\ 0 \\ 4 \end{pmatrix} - \frac{v_3 \cdot v_2}{v_2 \cdot v_2} v_2 = \begin{pmatrix} -2 \\ 1 \\ 2 \\ 1 \end{pmatrix} - \frac{6}{6} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix}.$$

Now divide by the length to obtain the following orthonormal basis

$$B = \{b_1, b_2, b_3\} = \{\frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 2 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}\}.$$

(f) (6 points). Compute the coordinate vectors of u_1, u_2, u_3 with respect to B; $[u_1]_B$, $[u_2]_B$, $[u_3]_B$.

Answer:

$$[u_1]_B = \begin{pmatrix} u_1 \cdot b_1 \\ u_1 \cdot b_2 \\ u_1 \cdot b_3 \end{pmatrix} = \begin{pmatrix} 5/\sqrt{5} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{5} \\ 0 \\ 0 \end{pmatrix}.$$

$$[u_2]_B = \begin{pmatrix} u_2 \cdot b_1 \\ u_2 \cdot b_2 \\ u_2 \cdot b_3 \end{pmatrix} = \begin{pmatrix} -5/\sqrt{5} \\ 6/\sqrt{6} \\ 0 \end{pmatrix} = \begin{pmatrix} -\sqrt{5} \\ \sqrt{6} \\ 0 \end{pmatrix}.$$

$$[u_3]_B = \begin{pmatrix} u_3 \cdot b_1 \\ u_3 \cdot b_2 \\ u_3 \cdot b_3 \end{pmatrix} = \begin{pmatrix} 10/\sqrt{5} \\ 6/\sqrt{6} \\ 2 \end{pmatrix} = \begin{pmatrix} 2\sqrt{5} \\ \sqrt{6} \\ 2 \end{pmatrix}.$$

(g) (4 points). Which of the following are in V (hint: use matrix B).

$$w_1 = \begin{pmatrix} -1 \\ 2 \\ -1 \\ 1 \end{pmatrix}, w_2 = \begin{pmatrix} 2 \\ -1 \\ 1 \\ -1 \end{pmatrix}, w_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 2 \end{pmatrix}.$$

Answer:
Use the hint: V is the null space of B. So w_1 is in V if and only if $Bw_1 = 0$. So calculate Bw_1, Bw_2 and Bw_3 and check which ones are zero. The result is that only $w_2 \in V$.
You could of course also row-reduce $(u_1 u_2 u_3 | w_1 w_2 w_3)$ and find out that of the three columns w_1 w_2 w_3 only the column for w_2 is consistent but this computation takes more time.

2. Let

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{pmatrix}$$

Compute the following:

(a) (4 points). The reduced row echelon form of A.

Answer:

$$\begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(b) (2 points). The rank of A.

Answer: rank is 2.

(c) (2 points). A basis for the column space $\mathcal{CS}(A)$ of A.

Answer: column 1 and column 2 of A, a basis is $\{u_1, u_2\}$ where

$$u_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix}.$$

(d) (4 points). An orthonormal basis for the column space.

$$v_1 = u_1$$

$$v_2 = u_2 - \frac{u_2 \cdot v_1}{v_1 \cdot v_1} v_1 = \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} - \frac{50}{30} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} = \frac{1}{3} \left(3 \begin{pmatrix} 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} - 5 \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} \right)$$

$$= \frac{1}{3} \begin{pmatrix} 4 \\ 2 \\ 0 \\ -2 \end{pmatrix} = \frac{2}{3} \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

Now divide by the lengths and you get

$$\frac{1}{\sqrt{30}} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \quad \frac{1}{\sqrt{6}} \begin{pmatrix} 2 \\ 1 \\ 0 \\ -1 \end{pmatrix}.$$

(e) (2 points). A basis for the null space $\mathcal{NS}(A)$ of A.

Answer: one column doesn't have a row-leader so there is one free variable. So the basis has just one element $(1/2 \ -3/2 \ 1)^T$. Note: this column vector is written as the transpose of a row vector to save some space on the paper.

(f) (2 points). A basis for the row space $\mathcal{RS}(A)$ of A.

Answer: $\{(1 \ 0 \ -1/2), \ \ (0 \ 1 \ 3/2)\}$.

(g) (3 points). Are the columns of A linearly dependent? If so, then give a linear relation.

Answer: The columns are dependent if and only if the number of columns is greater than the rank. So yes, they are dependent. $C_3 = -\frac{1}{2}C_1 + \frac{3}{2}C_2$, so $-\frac{1}{2}C_1 + \frac{3}{2}C_2 - C_3 = 0$.

(h) (3 points). Are the rows of A linearly dependent? If so, then give a linear relation.

Answer: The rows are dependent if and only if the number of rows is greater than the rank. So yes, they are dependent. How to find a linear relation? One way is to make the rows into columns (i.e. take A^T) and compute the null space. Each non-zero element of $NS(A^T)$ gives you a linear relation between the columns of A^T and so a linear relation between the rows of A. But there is a quicker way to find a linear relation, with some trial and error search we can see for example that $R_1 + R_4 = R_2 + R_3$ so we find the following linear relation $R_1 - R_2 - R_3 + R_4 = 0$. There are also many other linear relations. For example, $R_3 - R_2 = (1\ 1\ 1) = R_2 - R_1$. So $(R_2 - R_1) - (R_3 - R_2) = 0$, so $-R_1 + 2R_2 - R_3 = 0$.

3. Let

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_6 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) (4 points). Compute the reduced row echelon form of $B = (u_1\ u_2\ u_3\ u_4\ u_5\ u_6)$.

Answer:

$$\begin{pmatrix} 1 & 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 1 & 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

(b) (10 points). Compute a basis of the null space, a basis of the row space and a basis of the column space of B.

Answer: NS: $\{(-1/2\ 1/2\ 0\ 1\ 0\ 0)^T,\ (-1/2\ -1/2\ 1\ 0\ 0\ 0)^T\}$.
RS: $\{(1\ 0\ 1/2\ 1/2\ 0\ 0), (0\ 1\ 1/2\ -1/2\ 0\ 0), (0\ 0\ 0\ 0\ 1\ 0), (0\ 0\ 0\ 0\ 0\ 1)\}$.
CS: $\{u_1, u_2, u_5, u_6\}$. Notice that the column space is the span of these 4 linearly independent vectors in $\mathbb{R}^4$. Therefore, the column space is a subspace of $\mathbb{R}^4$ and has dimension 4, and so $CS(B) = \mathbb{R}^4$. So any basis of $\mathbb{R}^4$ (e.g. the standard basis) is a correct answer.

(c) (6 points). Give a basis for each of the following vector spaces:
SPAN($\{u_1\}$), a basis is $\{u_1\}$
SPAN($\{u_1, u_2\}$), a basis is $\{u_1, u_2\}$
SPAN($\{u_1, u_2, u_3\}$), a basis is $\{u_1, u_2\}$
SPAN($\{u_1, u_2, u_3, u_4\}$), a basis is $\{u_1, u_2\}$
SPAN($\{u_1, u_2, u_3, u_4, u_5\}$), a basis is $\{u_1, u_2, u_5\}$
SPAN($\{u_1, u_2, u_3, u_4, u_5, u_6\}$), a basis is $\{u_1, u_2, u_5, u_6.\}$

(d) (5 points). Whenever $\text{SPAN}(\{u_1, u_2, \ldots, u_n\}) = \text{SPAN}(\{u_1, u_2, \ldots, u_n, u_{n+1}\})$ express u_{n+1} as a linear combination of $u_1, u_2, \ldots, u_n$.

Answer: $u_3 = \frac{1}{2}u_1 + \frac{1}{2}u_2$, $u_4 = \frac{1}{2}u_1 - \frac{1}{2}u_2$.

4. Let $u_1 = x^3 - 4x$, $u_2 = x^2 - x - 2$, $u_3 = x^3 - x^2 - x - 2$, $u_4 = x^2 - 4$.
Let $v_1 = x - 2$, $v_2 = x^2 - 2x$, $v_3 = x^3 - 2x^2$.
Let $V = \text{SPAN}(\{u_1, u_2, u_3, u_4\})$.

(a) (6 points) V is a subspace of P_3 and the polynomials u_1, u_2, u_3, u_4, v_1, v_2, v_3 are elements of P_3. Give the coordinate vectors of these polynomials with respect to the basis $\{1, x, x^2, x^3\}$ of P_3.

Answer:

$$\begin{pmatrix} 0 \\ -4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -2 \\ 1 \end{pmatrix}.$$

(b) (10 points). Show that $B = \{v_1, v_2, v_3\}$ is a basis for V and compute $[u_1]_B$, $[u_2]_B$, $[u_3]_B$ and $[u_4]_B$.

Answer: Row-reduce $(v_1 v_2 v_3 | u_1 u_2 u_3 u_4)$. This will show that v_1, v_2, v_3 are linearly independent and that u_1, u_2, u_3, u_4 are in the SPAN of v_1, v_2, v_3. The result of this row-reduction is:

$$\begin{pmatrix} -2 & 0 & 0 & 0 & -2 & -2 & -4 \\ 1 & -2 & 0 & -4 & -1 & -1 & 0 \\ 0 & 1 & -2 & 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}.$$

So $V = \text{SPAN}(\{u_1, u_2, u_3, u_4\})$ is a subspace of $\text{SPAN}(\{v_1, v_2, v_3\})$ and the latter has basis $\{v_1, v_2, v_3\}$ and so it has dimension 3. In order to check that these two spaces are actually equal we have to check that the dimension of V is 3, in other words the rank of matrix $(u_1\ u_2\ u_3\ u_4)$ is 3. I'll skip that check.

(c) (3 points). Are u_1, u_2, u_3, u_4 linearly dependent or independent?

Answer: we have 4 vectors in a space that has $\{v_1, v_2, v_3\}$ as a basis, so 4 vectors in a three dimensional space. Hence: linearly dependent.

(d) (3 points). Is $\{f \in P_3 | f(2) = 0\}$ a subspace of P_3? If so, can you find a basis for it?

Notice that v_1, v_2, v_3 are elements of $\{f \in P_3 | f(2) = 0\}$. The vector space $\{f \in P_3 | f(2) = 0\}$ is a subspace of P_3 and is not equal to P_3. So its dimension is less than the dimension of P_3, so its dimension is < 4. But since we have three linearly independent elements v_1, v_2, v_3 in a space $\{f \in P_3 | f(2) = 0\}$ of dimension less than 4, it must be so that these three form a basis and that the dimension is 3. So we see that $V = \{f \in P_3 | f(2) = 0\}$.

5. (8 points). Let A be a 5 by 7 matrix for which the rank is 4. Compute the following:

- The dimension of the null space of A. Answer: $7 - 4 = 3$.
- The dimension of the column space of A. Answer: rank $= 4$.
- The dimension of the row space of A. Answer rank $= 4$.
- The dimension of the null space of A^T. Answer: the number of columns of A^T is 5 and the rank is the same as the rank of A. So the number of columns minus the rank is $5 - 4 = 1$.
- The dimension of the column space of A^T. rank $= 4$.
- The dimension of the row space of A^T. rank $= 4$.
- Are the rows of A linearly dependent or independent? Number of rows is smaller than the rank, hence dependent.
- Are the columns of A linearly dependent or independent? Number of columns is smaller than the rank, hence dependent.