Linear algebra, final

April 27, 2001

1. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$$

   Compute the following:

   (a) (3 points). The reduced row echelon form of A.

   (b) (2 points). The rank of A.

   (c) (5 points). A basis for the column space $\mathcal{CS}(A)$ of A.

   (d) (4 points). A matrix B whose nullspace is the column space of A, i.e. $\mathcal{NS}(B) = \mathcal{CS}(A)$.

   (e) (6 points). A basis for the null space $\mathcal{NS}(A)$ of A.

   (f) (6 points). An orthonormal basis for the null space $\mathcal{NS}(A)$ of A.

   (g) (3 points). The characteristic polynomial.

   (h) (4 points). The eigenvalues. Verify your answer by checking that the trace of A is the sum of the eigenvalues and that the determinant of A is the product of the eigenvalues.

   (i) (4 points). Compute the eigenvector(s) for each eigenvalue. Verify your answer by multiplying A with these eigenvectors.

   (j) (3 points). Compute (if it exists) an invertible matrix P for which $D = P^{-1}AP$ is a diagonal matrix. Give this diagonal matrix D as well.

2. Let V be the set of all solutions of the differential equation $f''(x)+f(x) = 0$. Let $B = \{\sin x, \cos x\}$ be a basis for V. Let $T : V \to V$ be the linear map such that $T(f) = f'$.

   (a) (1 point). Compute the image of each basis element under T, i.e. compute the derivative of $\sin x$ and of $\cos x$.

   (b) (4 points). Compute the coordinate vectors $[T(\sin x)]_B$ and $[T(\cos x)]_B$.

   (c) (4 points). Compute the matrix $A = [T]_{BB}$.

   (d) (3 points). $T^4(f) = f''''$. Compute $T^4(f)$ for each f in B.
   As a result of this computation, can you predict without doing matrix multiplications what the matrix A^4 is? (to collect the points you need to include some explanation, one or two short sentences can be enough).

3. Let V be the set of all polynomials of degree at most 2. Let $B = \{1, x, x^2\}$ be a basis of V. Let $T : V \to V$ be the linear map
$T(f) = (x^2 - 1)f' - 2(x + 1)f$.
If $f = ax^2 + bx + c$ then
$T(f) = (x^2-1)f'-2(x+1)f = (x^2-1)(2ax+b)-2(x+1)(ax^2+bx+c) = (-2a-b)x^2 + (-2a-2b-2c)x - b - 2c$.

   (a) (2 points). Compute $T(f)$ for each f in B.

   (b) (5 points). Compute $[T]_{BB}$.

   (c) (2 points). Suppose $f \in V$ and suppose that $[f]_B = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$. Compute $[T(f)]_B$ using the matrix $[T]_{BB}$ that you computed above.

   (d) (2 points). Let $B' = \{x^2 - 1, x^2 - x + 1, x^2 + x + 1\}$ be another basis of V. Compute $T(f)$ for each f in B'.

   (e) (2 points). Notice that every f in B' is an eigenvector of T. For each of these eigenvectors f in B', what is the corresponding eigenvalue?

   (f) (2 points). Use the previous question to find $[T]_{B'B'}$. Note: do not spend a lot of time on a question that is only worth a few points.

4. Let $B = \{u_1, u_2\}$ and $B' = \{v_1, v_2\}$ where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, u_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, v_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

(a) (3 points). Let

$$V = \text{NS}(1 \quad -1 \quad 1) = \{ \begin{pmatrix} x \\ y \\ z \end{pmatrix}, x - y + z = 0\}$$

Show that B is a basis of V.

(b) (2 points). Show that B' is a basis of V.

(c) (2 points). Is SPAN(B) equal to SPAN(B')? If so, why? If not, why not?

(d) (2 points). Compute $[u_1]_{B'}$ and $[u_2]_{B'}$.

(e) (2 points). Compute $[v_1]_B$ and $[v_2]_B$.

(f) (3 points). Compute matrix P, the B to B' change of basis matrix.

(g) (2 points). Compute matrix P^{-1}.

(h) (2 points). Let $w = 3v_1 + 5v_2$. Compute $[w]_{B'}$.

(i) (3 points). Compute $[w]_B$ using a change of basis matrix. Which change of basis matrix should we use, P or P^{-1}?

(j) (10 points). Let $T : V \to V$ be a linear map given by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} y \\ x \\ -z \end{pmatrix}$$

Compute $A = [T]_{BB}$ and $A' = [T]_{B'B'}$. Hint: these must be 2 by 2 matrices. Neither one needs to be diagonal.
To check correctness of your answer, compute the trace of A and A' (i.e. the sum of the diagonal, which is also equal to the sum of the eigenvalues).

(k) (3 points). Can you think of any reason why these two matrices should have the same trace?

(l) (3 points). Looking at the linear map T, can you think of any reason why A should be equal to A^{-1}?

Good luck!