Linear algebra, test 3

March 22, 2001

Write down your name and SSN.

1. (35 points) Let

\[
A = \begin{pmatrix}
1 & 1 & -1 & -1 \\
5 & 6 & -1 & 0 \\
0 & 1 & 4 & 5
\end{pmatrix}.
\]

(a) Compute the reduced row echelon form of \(A\) and the rank of \(A\).
(b) Compute a basis \(B\) for the column space \(CS(A)\).
(c) Compute the coordinate vector (with respect to \(B\)) of each column of \(A\).
(d) Compute a basis for the null space \(NS(A)\).
(e) Give a linear relation between the columns of \(A\).
(f) Find a matrix \(C\) such that \(NS(C) = CS(A)\).
(g) Find (if it exists) a non-zero vector \(v \in \mathbb{R}^3\) that is orthogonal (dot product 0) to every column of \(A\).

2. Let \(A\) be the matrix \(A = (1 \ 1 \ 1 \ 1)\). Let \(V = NS(A)\). Let

\[
u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix},
\quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix},
\quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix},
\quad u_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix},
\quad u_5 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix},
\quad u_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix},
\quad u_7 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}.
\]

(a) (5 points) Which (if any) of the vectors \(u_1, \ldots, u_7\) is NOT in \(V\)?
(b) (7 points) If \(M\) is a matrix with \(m\) rows, \(n\) columns, and rank \(r\), then give a formula for the dimension of the null space of \(M\).
(c) (8 points) Compute the rank of \(A\), and compute the dimension of \(V\) with this formula.
(d) (15 points) Find out which of the following sets are a basis for \(V\):

\[
\begin{align*}
\{ u_1 \} \\
\{ u_1, u_2 \} \\
\{ u_1, u_2, u_3 \} \\
\{ u_1, u_2, u_3, u_4 \} \\
\{ u_1, u_2, u_4 \} \\
\{ u_1, u_5 \} \\
\{ u_1, u_2, u_5 \} \\
\{ u_1, u_2, u_4, u_5 \} \\
\{ u_1, u_6 \} \\
\{ u_1, u_2, u_6 \} \\
\{ u_1, u_2, u_4, u_5, u_6 \} \\
\{ u_1, u_2, u_3, u_4, u_5, u_6, u_7 \}.
\end{align*}
\]
(e) (5 points) Which of these sets are not a basis of \( V \) but just a spanning set for \( V \)?

(f) (5 points) Which of these sets are not a basis but are still linearly independent?

(g) (5 points) Let \( w \) be the vector

\[
\begin{pmatrix}
    x \\
    x^2 \\
    x \\
    x + 2
\end{pmatrix}
\]

where \( x \) is some unknown number. For which values of \( x \) would \( w \) be an element of \( V \)?

3. Let \( V \) be a subspace of \( \mathbb{R}^5 \) with as basis \( B = \{u_1, u_2\} \) where

\[
\begin{align*}
    u_1 &= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \\
    u_2 &= \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}.
\end{align*}
\]

Let

\[
\begin{align*}
    v_1 &= \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, \\
    v_2 &= \begin{pmatrix} 3 \\ 5 \\ 7 \\ 9 \\ 11 \end{pmatrix}, \\
    v_3 &= \begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \\ 11 \end{pmatrix}.
\end{align*}
\]

(a) (10 points) Is \( v_1 \in V \)? If so, then calculate \([v_1]_B\).

Same question for \( v_2 \) and \( v_3 \).

(b) (5 points) Let \( W = \text{SPAN}(v_1, v_2, v_3) \).

Is \( V \subseteq W \)?

Is \( W \subseteq V \)?

Is \( V = W \)?

Good luck!