Linear algebra, test 3 answers

1. Let

$$A = \begin{pmatrix} 1 & 1 & -1 & -1 \\ 5 & 6 & -1 & 0 \\ 0 & 1 & 4 & 5 \end{pmatrix}.$$

(a) Compute the reduced row echelon form of A and the rank of A.

Answer:

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & -5 & -6 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

$\text{rank}(A)$ = number of non-zero rows in $\text{rref}(A) = 2$

(b) Compute a basis B for the column space $CS(A)$.

Answer: As you can see in $\text{rref}(A)$, the columns Col3 and Col4 are linear combinations of Col1 and Col2, and Col1,Col2 are independent. So we only need Col1,Col2, because the other Columns are combinations of Col1,Col2.
A basis is:

$$B = \{\text{Col}_1(A), \text{Col}_2(A)\} = \left\{ \begin{pmatrix} 1 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 6 \\ 1 \end{pmatrix} \right\}.$$

(c) Compute the coordinate vector (with respect to B) of each column of A.

Answer: You can see from $\text{rref}(A)$ that Col1,Col2,Col3,Col4 are linear combinations of Col1,Col2, and you can also see what those combinations are:

$$[\text{Col}_1(A)]_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, [\text{Col}_2(A)]_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, [\text{Col}_3(A)]_B = \begin{pmatrix} -5 \\ 4 \end{pmatrix}, [\text{Col}_4(A)]_B = \begin{pmatrix} -6 \\ 5 \end{pmatrix}$$

(d) Compute a basis for the null space $\mathcal{NS}(A)$.

Answer: General solution of $AX = 0$ is:

$$X = \begin{pmatrix} 5x_3 + 6x_4 \\ -4x_3 - 5x_4 \\ x_3 \\ x_4 \end{pmatrix}$$

Basis is (plug in: one variable = 1, all other variables = 0, and do this for each free variable, and you get):

$$\left\{ \begin{pmatrix} 5 \\ -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ -5 \\ 0 \\ 1 \end{pmatrix} \right\}$$

(e) Give a linear relation between the columns of A.

Answer: We've already seen that for example Col3=-5Col1 + 4Col2, and that's a linear relation, so we're already done with this question. You can also write that as: 5Col1 -4Col2 + 1Col3 + 0Col4 = 0. Note that these numbers (5, -4, 1, 0) correspond to an element of the Nullspace (see previous question).

(f) Find a matrix C such that $NS(C) = CS(A)$.

Answer: Row-reduce the matrix $(A|K)$, then K is in $CS(A)$ when the system $(A|K)$ is consistent. But $(A|K)$ consistent is equivalent to some linear equations (namely: whenever rref(A) has a zero row, the corresponding entry on the right must be 0, and that "must be 0" is a linear equation in the entries of K. So for each zero row of rref(A) you get a linear equation for the entries of K). Those linear equations can be written in matrix form as $CK = 0$, where we have one row of C for each equation we found for the entries of K. Now $CK = 0$ is the same as saying $K \in NS(C)$. So we have to find the equations coming from "$(A|K)$ consistent", and write them down as a matrix equation $CK = 0$. Then that matrix C is the answer. If $K = (k_1\ k_2\ k_3)^T$ (T stands for transpose, to turn it into a column) then $(A|K)$ rowreduces to

$$\begin{pmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & k_3 - k_2 + 5k_1 \end{pmatrix}$$

where $*$ refers to entries that do not matter. So $K \in CS(A)$ if and only if $(A|K)$ consistent, if and only if $k_3 - k_2 + 5k_1 = 0$. Now $k_3 - k_2 + 5k_1$ equals CK if we set $C = (5\ \ -1\ 1)$. So $k_3 - k_2 + 5k_1 = 0$ if and only of $CK = 0$, if and only if $K \in NS(C)$. So $NS(C) = CS(A)$.

(g) Find (if it exists) a non-zero vector $v \in \mathbb{R}^3$ that is orthogonal (dotproduct 0) to every column of A.

Answer: $CK = 0$ for any K in the column space of A. So $CK = 0$ for any column K of A. Now

$$0 = CK = (5\ \ -1\ 1)K = C^T \cdot K$$

So

$$C^T = \begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$$

is orthogonal to every column of A.
NOTE: If C would have more than one row, then if we took the transpose of each row, all of those vectors that we would obtain in that way would be orthogonal to $CS(A)$, i.e. orthogonal to all columns of A.

2. Let A be the matrix $A = (1\ 1\ 1\ 1)$. Let $V = NS(A)$. Let

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, u_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, u_5 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$$

$$u_6 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, u_7 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

(a) Which (if any) of the vectors $u_1, \ldots, u_7$ is NOT in V?

Answer: A vector v is in $NS(A)$ if and only if $Av = 0$. So for each of those 7 vectors $v = u_1$, up to $v = u_7$, calculate: $Av = (1\ 1\ 1\ 1)v$ =sum of the entries of v. If $Av = 0$ then $v \in NS(A) = V$, and otherwise it is not in V.
If you do this calculation for all 7, you quickly find out that u_7 is not only one that's NOT in V.

(b) If M is a matrix with m rows, n columns, and rank r, then give a formula for the dimension of the null space of M.

Answer: $n - r$ (#cols - rank)

(c) Compute the rank of A, and compute the dimension of V with this formula.

Answer: A has only 1 row, and it is non-zero. So rank=1. The number of columns is 4. So we have dim(V)=4-1=3.

(d) Find out which of the following sets are a basis for V:
$\{u_1\}$
$\{u_1, u_2\}$
$\{u_1, u_2, u_3\}$
$\{u_1, u_2, u_3, u_4\}$
$\{u_1, u_2, u_4\}$
$\{u_1, u_5\}$
$\{u_1, u_2, u_5\}$
$\{u_1, u_2, u_4, u_5\}$
$\{u_1, u_6\}$
$\{u_1, u_2, u_6\}$
$\{u_1, u_2, u_4, u_5, u_6\}$.
$\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$.

Answer: Since dim(V)=3, any basis of V must contain 3 elements. That leaves only 4 candidates. The first candidate: $\{u_1, u_2, u_3\}$ is dependent (u_1, u_2 is independent but u_1, u_2, u_3 is dependent, see: $2u_3 = u_1 + u_2$). The second candidate $\{u_1, u_2, u_4\}$ is also dependent ($2u_4 = u_1 - u_2$). But u_5 is not in SPAN$\{u_1, u_2\}$ (easy to check) so the third candidate $\{u_1, u_2, u_5\}$ is independent and therefore a basis. The fourth candidate $\{u_1, u_2, u_6\}$ is also independent, and therefore a basis.

(e) (5 points) Which of these sets are not a basis of V but just a spanning set for V?

Answer: You must remember the following rules:
(1) S is a basis of V if and only if S is independent, and it is a spanning set of V
(2) if $k < n$ then S is not a spanning set of V
(3) if S contains vectors that are not in V then S is not a spanning set
(4) if $k = n$ and $S \subset V$ then S is a spanning set if and only if it is a basis.
(5) if $k = n$ and $S \subset V$ then S is independent if and only if it is a basis.
(6) if $k > n$ and $S \subset V$ then S is dependent.
(7) if $S \subset V$ then S is a spanning set if and only if it contains n linearly independent elements.
The last set $\{u_1, u_2, u_3, u_4, u_5, u_6, u_7\}$ is not a spanning set because it contains a vector that is not in V. We've already checked all sets with 3 elements (because of (4) and (5) we see that sets with 3 elements are independent if and only if they are a spanning set, if and only if they are a basis. This works whenever a set has the right number of elements, which in this case is 3, and those elements must be in V.). So the sets $\{u_1, u_2, u_3\}$ and $\{u_1, u_2, u_4\}$ are not spanning sets (their SPAN is only 2-dimensional because their SPAN is the same as the SPAN of $\{u_1, u_2\}$). Now any set with less than dim(V) elements can never be a spanning set, so we don't need to look at those. The candidates that remain are: $\{u_1, u_2, u_3, u_4\}$, $\{u_1, u_2, u_4, u_5\}$, $\{u_1, u_2, u_4, u_5, u_6\}$. The second and third set contain 3 independent vectors and therefore they are spanning sets (that's rule (7), this rule is true because the dimension of their span is at least 3 (hence equal to 3 because it's a subspace of V so the dimension can not be greater than 3)). But the first one does not contain 3 independent vectors (u_3, u_4 are both in the SPAN of u_1, u_2) so its SPAN is not big enough (its SPAN has only dimension 2 because we only have two independent vectors.

(f) (5 points) Which of these sets are not a basis but are still linearly independent?

Answer: We've already checked the sets with 3 elements. Any set with more than dim(V) elements in V is automatically dependent. So the only possibilities that are left are: $\{u_1\}$, $\{u_1, u_2\}$, $\{u_1, u_5\}$, $\{u_1, u_6\}$. It's easy to see that each of these is linearly independent. For example, $\{u_1\}$

is independent because it contains only 1 element which is not zero. And $\{u_1, u_6\}$ is independent because $u_1 \neq 0$ and u_6 is not a scalar multiple of u_1 (i.e. u_6 is not in the SPAN of u_1).

(g) (5 points) Let w be the vector

$$w = \begin{pmatrix} x \\ x^2 \\ x \\ x+2 \end{pmatrix}$$

where x is some unknown number. For which values of x would w be an element of V?

Answer: Remember that $V = NS(A)$ and that $w \in NS(A)$ if and only if $Aw = 0$. Now $Aw = x + x^2 + x + (x+2) = x^2 + 3x + 2 = (x+1)(x+2)$. So that's 0 when $x = -1$ or $x = -2$.

3. Let V be a subspace of $\mathbb{R}^5$ with as basis $B = \{u_1, u_2\}$ where

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, u_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{pmatrix}.$$

Let

$$v_1 = \begin{pmatrix} 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} 3 \\ 5 \\ 7 \\ 9 \\ 11 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 3 \\ 5 \\ 7 \\ 11 \end{pmatrix}$$

(a) Is $v_1 \in V$? If so, then calculate $[v_1]_B$.
Same question for v_2 and v_3.

Answer: Row-reduce $(B|v_1 v_2 v_3) = (u_1 u_2 | v_1 v_2 v_3)$ and you get:

$$\begin{pmatrix} 1 & 0 & 6 & 1 & * \\ 0 & 1 & -1 & 2 & * \\ 0 & 0 & 0 & 0 & \neq 0 \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & * \end{pmatrix}$$

where as usual $*$ denotes entries whose values are irrelevant. As you can see, the last column v_3 is not consistent, so that's not in the span of u_1, u_2. As for the other ones, we can read of the coordinate vector

$$[v_1]_B = \begin{pmatrix} 6 \\ -1 \end{pmatrix}, [v_2]_B = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(b) Let $W = \text{SPAN}(v_1, v_2, v_3)$.
Is $V \subseteq W$?
Is $W \subseteq V$?
Is $V = W$?

Answer: $V \subseteq W$? Now $v_1, v_2 \in W$ and so $\text{SPAN}(v_1, v_2) \subseteq W$. But v_1, v_2 are two independent elements of V and dim(V)=2. So: dim(V) independent elements of V, that implies: v_1, v_2 is a basis of V. So $\text{SPAN}(v_1, v_2) = V$. And we've seen that that is a subspace of W. So the answer is yes.

Is $W \subseteq V$? Now since W is the SPAN of v_1, v_2, v_3, this is the same as asking: are all three v_1, v_2, v_3 in V? The answer is no because v_3 is not in V.

Is $V = W$? This is the same thing as asking: are $V \subseteq W$ and $W \subseteq V$ both true? The answer is no.