Linear algebra, test 3, 11:00-12:15

April 15, 2003

1. Let
$$A = \begin{pmatrix} 6 & -1 & 0 \\ 0 & 2 & 2 \\ 6 & 0 & 1 \end{pmatrix}.$$

   (a) Give the characteristic polynomial of A.

   (b) Compute the eigenvalues (hint: these turn out to be whole numbers) of A and for each eigenvalue compute a corresponding eigenvector (show computation).

   (c) Give a matrix P such that $P^{-1}AP$ is a diagonal matrix (no computation necessary).

2. Let V be the subspace of $\mathbb{R}^4$ consisting of those vectors for which the sum of the entries is zero. Let $B = b_1, b_2, b_3$ be a basis of V, where

$$b_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}, b_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, b_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}.$$

and let $C = c_1, c_2, c_3$ be another basis of V, where

$$c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

(a) Let $T : V \to V$ be a linear map defined by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} x_4 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

Give the matrix $[T]_B$ of the linear map T with respect to the basis B of V.

(b) Compute the matrix $A = [T]_C$. Looking at the linear map T, can you predict without matrix multiplication what the matrix A^4 will be?

(c) Give the characteristic polynomial of T.

(d) Give the change of basis matrix from B to C.

(e) Give the change of basis matrix from C to B.

(f) If $S : V \to V$ is any linear map, then write down a matrix P such that $[S]_C = P^{-1} \, [S]_B \, P$.

3. Let V be a vector space of dimension 2, and let $B = u_1, u_2$ be a basis of V. Let $T : V \to V$ be the linear map for which $T(u_1) = u_1 + 2u_2$, and $T(u_2) = u_1$.

(a) Give the matrix $[T]_B$.

(b) Compute the characteristic polynomial of matrix $[T]_B$.

(c) Compute the eigenvectors of matrix $[T]_B$.

(d) Give the eigenvectors of linear map T (not of the matrix $[T]_B$).

(e) (no computation necessary) Give a basis C of V such that $[T]_C$ is a diagonal matrix.