Linear algebra, sample questions

April 7, 2003

1. Let

\[
A = \begin{pmatrix}
-2 & 1 & 0 \\
-2 & 1 & 0 \\
-3 & 1 & 1
\end{pmatrix}
\]

Compute the eigenvalues of this matrix. For each eigenvalue, compute an eigenvector.

2. Let \( T: \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear map given by

\[
T \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}
\]

Let \( B = \{e_1, e_2\} \) and \( B' = \{v_1, v_2\} \) where

\[
e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

(a) Give the \( B \) to \( B' \) change-of-basis matrix and the \( B' \) to \( B \) change-of-basis matrix.

(b) Compute the following

i. \([T]_B\)

ii. \([T]_{B'}\)

3. Let

\[
A = \begin{pmatrix}
8 & -10 \\
5 & -7
\end{pmatrix}.
\]

(a) Compute the eigenvalues of \( A \) and for each eigenvalue compute one corresponding eigenvector.

(b) Compute a matrix \( P \) such that \( P^{-1}AP \) is a diagonal matrix.
4. Let $B = \{u_1, u_2\}$ where $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and let $B' = \{v_1, v_2\}$. Let $T$ be a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, and $T(v_1) = 2v_1$, $T(v_2) = 3v_2$.

(a) Compute $[u_1]_{B'}$ and $[u_2]_{B'}$.
(b) Compute $T(u_1)$ and $T(u_2)$.
(c) Compute $A = [T]_{B'}$.
(d) Find $A' = [T]_{B'}$ and the eigenvalues of $A$.

5. Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

and let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $A$:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$ 

Let $B = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and let $B' = \{u_1, u_2, u_3\}$ where

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$ 

(a) Is $B'$ a basis for $\mathbb{R}^3$?
(b) Compute the change of basis matrix from $B$ (old basis) to $B'$ (new basis) and the change of basis matrix from $B'$ to $B'$.
(c) Use the change of basis matrix to compute

$$\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}_{B'}, \quad \begin{pmatrix} -2 \\ 2 \end{pmatrix}_{B'}, \quad \text{and} \quad \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}_{B'}.$$

(d) Compute $[T]_{B'}$. Hint: if you made no computation errors the result should be an upper triangular matrix.
(e) What are the eigenvalues of $A$?

6. Consider the following subspace $V$ of $\mathbb{R}^3$

$V = \text{SPAN}(v_1, v_2, v_3)$ where

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$
(a) Show that \( B = \{v_1, v_2\} \) is a basis of \( V \), and compute \([v_3]_B\).

(b) For which real number \( x \) is the following vector \( w \) an element of \( V \)?

\[
    w = \begin{pmatrix}
        2 \\
        x \\
        x - 1
    \end{pmatrix}
\]

Write \( w \) as a linear combination of \( \{v_1, v_2\} \).

(c) Let \( T : V \to V \) be a linear map defined by

\[
    T \begin{pmatrix}
        x \\
        y \\
        z
    \end{pmatrix} = \begin{pmatrix}
        x - y \\
        0 \\
        z - y
    \end{pmatrix}.
\]

Give the matrix \([T]_B\) of the linear map \( T \) with respect to the basis \( B \) of \( V \). Hint: \( \dim(V) = 2 \) so this must be a 2 by 2 matrix.

(d) Compute: the rank of \( T \), the dimension of the Nullspace of \( T \), and compute \([T^2]_B\).