Linear algebra, sample questions

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1. Let
$$A = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix}$$
Compute the eigenvalues of this matrix. For each eigenvalue, compute an eigenvector.

2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by
$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}$$
Let $B = \{e_1, e_2\}$ and $B' = \{v_1, v_2\}$ where
$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$
   (a) Give the B to B' change-of-basis matrix and the B' to B change-of-basis matrix.
   (b) Compute the following
      i. $[T]_B$
      ii. $[T]_{B'}$

3. Let
$$A = \begin{pmatrix} 8 & -10 \\ 5 & -7 \end{pmatrix}.$$
   (a) Compute the eigenvalues of A and for each eigenvalue compute one corresponding eigenvector.
   (b) Compute a matrix P such that $P^{-1}AP$ is a diagonal matrix.

4. Let $B = \{u_1, u_2\}$ where $u_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, and let $B' = \{v_1, v_2\}$. Let T be a linear map from $\mathbb{R}^2$ to $\mathbb{R}^2$, and $T(v_1) = 2v_1$, $T(v_2) = 3v_2$.

   (a) Compute $[u_1]_{B'}$ and $[u_2]_{B'}$.
   (b) Compute $T(u_1)$ and $T(u_2)$.
   (c) Compute $A = [T]_B$.
   (d) Find $A' = [T]_{B'}$ and the eigenvalues of A.

5. Let

$$A = \begin{pmatrix} 2 & 1 & 1 \\ -1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}$$

and let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by A:

$$T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Let $B = \{e_1, e_2, e_3\}$ be the standard basis of $\mathbb{R}^3$ and let $B' = \{u_1, u_2, u_3\}$ where

$$u_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

   (a) Is B' a basis for $\mathbb{R}^3$?
   (b) Compute the change of basis matrix from B (old basis) to B' (new basis) and the change of basis matrix from B' to B'.
   (c) Use the change of basis matrix to compute

$$\left[\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}\right]_{B'}, \quad \left[\begin{pmatrix} 2 \\ -2 \\ 2 \end{pmatrix}\right]_{B'}, \quad \text{and} \quad \left[\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right]_{B'}$$

   (d) Compute $[T]_{B'}$. Hint: if you made no computation errors the result should be an upper triangular matrix.
   (e) What are the eigenvalues of A?

6. Consider the following subspace V of $\mathbb{R}^3$
$V = \text{SPAN}(v_1, v_2, v_3)$ where

$$v_1 = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \quad v_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad v_3 = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}.$$

(a) Show that $B = \{v_1, v_2\}$ is a basis of V, and compute $[v_3]_B$.

(b) For which real number x is the following vector w an element of V?

$$w = \begin{pmatrix} 2 \\ x \\ x-1 \end{pmatrix}$$

Write w as a linear combination of $\{v_1, v_2\}$.

(c) Let $T : V \to V$ be a linear map defined by

$$T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x-y \\ 0 \\ z-y \end{pmatrix}.$$

Give the matrix $[T]_B$ of the linear map T with respect to the basis B of V. Hint: $\dim(V) = 2$ so this must be a 2 by 2 matrix.

(d) Compute: the rank of T, the dimension of the Nullspace of T, and compute $[T^2]_B$.