Linear algebra, sample questions.

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1.

$$A = \begin{pmatrix} -1 & 2 \\ 2 & -5 \end{pmatrix}$$

(a) Write A as a product of elementary matrices.

(b) Compute A^{-1}

(c) Suppose that $AX = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. What is X?

2.

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 3 & -2 \\ -1 & 0 & 1 \end{pmatrix}$$

(a) Compute an LU factorization of A.

(b) Use the LU decomposition to solve the following

$$AX = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

3. Let

$$A = \begin{pmatrix} 1 & -2 & 2 \\ 2 & 3 & -2 \\ 0 & 1 & -1 \end{pmatrix}$$

(a) Compute the inverse of A.

(b) Find a 3 by 2 matrix B such that

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

4. True or false?

(a) If A is row equivalent to B then the following two systems have the same solutions:

$$AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad BX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

(b) If A is a square matrix and if A^2 is the zero matrix (all entries are zero) then A is also the zero matrix.

(c) If A is a square matrix then A is row equivalent to the identity matrix if and only if

$$AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

has only one solution (the trivial solution).

(d) If A and B are square matrices, and if AB is invertible, then A, B are both invertible.

(e) If T is a one-to-one linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ then T is also onto.

(f) If T is a linear map from $\mathbb{R}^3$ to $\mathbb{R}^5$ then T is not one-to-one.

5. Let

$$u = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}$$

(a) Let $A = (u \;\; v)$. Compute a matrix B such that the null space of B is the column space of A.

(b) For which values of x and y is the vector

$$w = \begin{pmatrix} x+1 \\ x \\ y \\ -y \end{pmatrix}$$

an element of $\text{SPAN}(\{u, v\})$?

6. Let

$$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{pmatrix}$$

Compute the following:

(a) The reduced row echelon form of A.

(b) The rank of A.

(c) A basis for the column space of A.

(d) A basis for the null space of A.

(e) Are the columns of A linearly dependent? If so, then give a linear relation.

7. Let

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_6 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Compute the reduced row echelon form of $B = (u_1 \; u_2 \; u_3 \; u_4 \; u_5 \; u_6)$.

(b) Compute a basis of the null space, a basis of the row space and a basis of the column space of B.

(c) Give a basis for each of the following vector spaces:
$\text{SPAN}(\{u_1\})$, $\text{SPAN}(\{u_1, u_2\})$, $\text{SPAN}(\{u_1, u_2, u_3\})$, $\text{SPAN}(\{u_1, u_2, u_3, u_4\})$, $\text{SPAN}(\{u_1, u_2, u_3, u_4, u_5\})$
, $\text{SPAN}(\{u_1, u_2, u_3, u_4, u_5, u_6\})$.

(d) Whenever $\text{SPAN}(\{u_1, u_2, \ldots, u_n\}) = \text{SPAN}(\{u_1, u_2, \ldots, u_n, u_{n+1}\})$ express u_{n+1} as a linear combination of $u_1, u_2, \ldots, u_n$.

8. Let A be a 5 by 7 matrix for which the rank is 4. Compute the following:

- The dimension of the null space of A.
- The dimension of the column space of A.
- The dimension of the row space of A.
- The dimension of the null space of A^T.
- The dimension of the column space of A^T.
- The dimension of the row space of A^T.
- Are the columns of A linearly dependent or independent?
- Are the columns of A^T linearly dependent or independent?