Linear Algebra, Final, Thursday May 1, 2003, 12:30 - 2:30
WRITE DOWN YOUR NAME:

1. Let A be a 7 by 8 matrix for which the rank is 5.

   - (1 point). What is the dimension of the Null Space of A.
   - (1 point). What is the dimension of the Column Space of A.
   - (1 point). Are the columns of A linearly dependent or independent?
   - (1 point). Is the linear map given by A one-to-one? (yes/no).
   - (1 point). Is the linear map given by A onto? (yes/no).

2. (5 points). Write $A = \begin{pmatrix} 1 & 0 \\ 3 & 2 \end{pmatrix}$ as a product of elementary matrices.

3. (10 points). Take the following basis $B = f_1, f_2, f_3$ of the vector space V of all polynomials of degree ≤ 2, where $f_1 = 1$, $f_2 = 1+x$, and $f_3 = 1+x+x^2$. Let $T : V \to V$ be the linear map given by differentiation. What is $[T]_B$?

4. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ and compute:

(a) (2 points). The reduced row echelon form of A.

(b) (2 points). The rank of A.

(c) (2 points). A basis for the column space of A.

(d) (2 points). A basis for the null space of A.

(e) (2 points). Can matrix A be written as a product of elementary matrices? (yes/no).

(f) (10 points). The characteristic polynomial is $\lambda^2(\lambda - 2)$. For each eigenvalue, compute corresponding eigenvectors, and give a matrix P such that $P^{-1}AP = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

5. (10 points). Let V be a vector space with the following basis:

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad u_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \\ 0 \end{pmatrix} \quad u_3 = \begin{pmatrix} 0 \\ 0 \\ 3 \\ 3 \end{pmatrix}.$$

Apply Gram Schmidt to obtain an orthogonal basis v_1, v_2, v_3 of V.

6. Let $V = \{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid x_1 + x_2 + x_3 = 0\}$. Let $T : V \to V$ be given by

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 5x_2 - 7x_3 \\ 3x_1 + 7x_3 \\ -3x_1 - 5x_2 \end{pmatrix}$$

Problem: Find a basis C of V such that $[T]_C$ is diagonal.

Before answering this Problem, first do the following questions

(a) (2 points). Explain why V has dimension 2.

(b) (1 point). Write down a vector in $\mathbb{R}^3$ such that: Not all three entries are 0, and the sum of the entries is 0.
Then write down another vector in $\mathbb{R}^3$, again, not all entries 0 and the sum of the entries must be 0, but this vector must not be a scalar multiple of the previous one.
Then let B be the these two vectors.

(c) (2 points). Explain why B must be a basis of V, for every possible choice you could have made in the previous question.

(d) (2 points). Apply T to the elements of your basis B. To check for computation errors, check if your result is actually in V.

(e) (4 points). Compute matrix $[T]_B$.

(f) (5 points). Compute the eigenvalues of matrix $[T]_B$. Hint: If you get square roots then first double-check your answer on the previous two questions and then on this question.

(g) (5 points). For each eigenvalue compute an eigenvector of $[T]_B$.

(h) (4 points). Use those eigenvectors to find eigenvectors of T.

(i) (1 point). Give an answer to the Problem.

7. Suppose $B = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\}$ is the basis of some vector space V (so V is a two-dimensional subspace of $\mathbb{R}^3$).

(a) (4 points). Let $u \in V$ and suppose $[u]_B = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$. What is u?

(b) (4 points). Let $v = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$. Compute $[v]_B$.

(c) (4 points). Suppose that C is another basis, and that the B to C change of basis matrix is

$$\underset{C \leftarrow B}{P} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$$

What is $[v]_C$?

(d) (4 points). Compute the matrix $\underset{B \leftarrow C}{P}$.

(e) (4 points). If $w \in V$ and $[w]_C = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Then what is $[w]_B$ and what is w?

(f) (4 points). What is basis C?