1. Let

$$A = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix}$$

Compute the following:

(a) The reduced row echelon form of A.

(b) The rank of A.

(c) A basis for the column space $\mathcal{CS}(A)$ of A.

(d) A basis for the null space $\mathcal{NS}(A)$ of A.

(e) An orthonormal basis for the row space $\mathcal{RS}(A)$ of A.

(f) Give a linear relation between the rows of A, and give a linear relation between the columns of A.

(g) The characteristic polynomial.

(h) The eigenvalues. Verify your answer by checking that the trace of A is the sum of the eigenvalues and that the determinant of A is the product of the eigenvalues.

(i) Compute an eigenvector for each eigenvalue. Verify your answer by multiplying A with these eigenvectors. Are these eigenvectors linearly independent?

2. Consider the following subspace V of $\mathbb{R}^3$
$V = \text{SPAN}(v_1, v_2, v_3)$ where

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Show that $B = \{v_1, v_2\}$ is a basis of V.

(b) For which real number x is the following vector w an element of V?

$$w = \begin{pmatrix} 2 \\ 3 \\ x \end{pmatrix}$$

Write w as a linear combination of $\{v_1, v_2\}$.

(c) Let

$$u_1 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad u_2 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad u_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

Show that u_1, u_2, u_3 are elements of V. Give the coordinates of u_1, u_2, u_3 with respect to the basis B.

(d) Let $T : V \to V$ be a linear map defined by

$$T(v_1) = u_1, \quad T(v_2) = u_2.$$

Give the matrix $[T]_{BB}$ of the linear map T with respect to the basis B of V. Hint: $\dim(V) = 2$ so this must be a 2 by 2 matrix.

(e) Compute: the rank and the nullity of T. If T is invertible, then compute the matrix of the inverse map T^{-1} with respect to the basis B.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}$$

where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Let $B = \{e_1, e_2\}$ and $B' = \{u_1, u_2\}$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(a) Compute A^{-1}, $\det(A)$ and $\det(A^{-1})$.

(b) Compute the eigenvalues of A.

(c) Give the B to B' change-of-basis matrix and the B' to B change-of-basis matrix.

(d) Compute the following

i. $[T]_{BB}$

ii. $[T]_{BB'}$

iii. $[T]_{B'B}$

iv. $[T]_{B'B'}$