1. Let

$$A = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix}$$

Compute the following:

(a) The reduced row echelon form of $A$.
(b) The rank of $A$.
(c) A basis for the column space $\mathcal{C}(A)$ of $A$.
(d) A basis for the null space $\mathcal{N}(A)$ of $A$.
(e) An orthonormal basis for the row space $\mathcal{R}(A)$ of $A$.
(f) Give a linear relation between the rows of $A$, and give a linear relation between the columns of $A$.
(g) The characteristic polynomial.
(h) The eigenvalues. Verify your answer by checking that the trace of $A$ is the sum of the eigenvalues and that the determinant of $A$ is the product of the eigenvalues.
(i) Compute an eigenvector for each eigenvalue. Verify your answer by multiplying $A$ with these eigenvectors. Are these eigenvectors linearly independent?

2. Consider the following subspace $V$ of $\mathbb{R}^3$

$V = \text{SPAN}(v_1, v_2, v_3)$ where

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Show that $B = \{v_1, v_2\}$ is a basis of $V$.
(b) For which real number $x$ is the following vector $w$ an element of $V$?

$$w = \begin{pmatrix} 2 \\ 3 \\ x \end{pmatrix}$$

Write $w$ as a linear combination of $\{v_1, v_2\}$.

(c) Let

$$u_1 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad u_2 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad u_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

Show that $u_1, u_2, u_3$ are elements of $V$. Give the coordinates of $u_1, u_2, u_3$ with respect to the basis $B$. 

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(d) Let $T : V \to V$ be a linear map defined by

$$T(v_1) = u_1, \quad T(v_2) = u_2.$$ 

Give the matrix $[T]_{BB}$ of the linear map $T$ with respect to the basis $B$ of $V$. Hint: $\dim(V) = 2$ so this must be a 2 by 2 matrix.

(e) Compute: the rank and the nullity of $T$. If $T$ is invertible, then compute the matrix of the inverse map $T^{-1}$ with respect to the basis $B$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by

$$T \begin{pmatrix} x \\ y \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}$$

where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$ 

Let $B = \{e_1, e_2\}$ and $B' = \{u_1, u_2\}$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$ 

(a) Compute $A^{-1}$, $\det(A)$ and $\det(A^{-1})$.

(b) Compute the eigenvalues of $A$.

(c) Give the $B$ to $B'$ change-of-basis matrix and the $B'$ to $B$ change-of-basis matrix.

(d) Compute the following

i. $[T]_{BB}$

ii. $[T]_{BB'}$

iii. $[T]_{B'B}$

iv. $[T]_{B'B'}$