1. Let

$$A = \begin{pmatrix} -2 & 1 & 0 \\ -2 & 1 & 0 \\ -3 & 1 & 1 \end{pmatrix}$$

Compute the following:

(a) The reduced row echelon form of A.

$$\text{rref}(A) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

(b) The rank of A.
rank$(A) = 2$.

(c) A basis for the column space $\mathcal{CS}(A)$ of A.
From reduced row echolon form one can see that the following:

$$\{\text{Col}_1(A), \text{Col}_2(A)\} = \{\begin{pmatrix} -2 \\ -2 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\}$$

is a basis of the column space. Alternatively, by computing the reduced column echelon form of A one finds the basis

$$\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\}.$$

(d) A basis for the null space $\mathcal{NS}(A)$ of A.

$$\{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}\}$$

(e) An orthonormal basis for the row space $\mathcal{RS}(A)$ of A.
Apply Gram-Schmidt on

$$\{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ -2 \end{pmatrix}\}.$$

Result is

$$\{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \frac{1}{\sqrt{3}}\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}\}.$$

(f) Give a linear relation between the rows of A, and give a linear relation between the columns of A.
$\text{Row}_1 - \text{Row}_2 = 0$ and $\text{Col}_1 + 2\text{Col}_2 + \text{Col}_3 = 0$.

(g) The characteristic polynomial.
$\lambda^3 - \lambda$.

(h) The eigenvalues. Verify your answer by checking that the trace of A is the sum of the eigenvalues and that the determinant of A is the product of the eigenvalues.
Eigenvalues: $\lambda_1 = 0$, $\lambda_2 = 1$, and $\lambda_3 = -1$.
Sum eigenvalues = 0. Trace = sum(diagonal) = 0. OK.
Product eigenvalues = 0. Det(A) = 0. OK.

(i) Compute an eigenvector for each eigenvalue. Verify your answer by multiplying A with these eigenvectors. Are these eigenvectors linearly independent?
Eigenvalue 0: $v_1 = (1, 2, 1)^T$ (from 1d).
Eigenvalue 1: nullspace of $I - A$, result: $v_2 = (0, 0, 1)^T$.
Eigenvalue -1: nullspace of $-I - A$, result: $v_3 = (1, 1, 1)^T$.
$Av_1 = \lambda_1 \cdot v_1$. $Av_2 = \lambda_2 \cdot v_2$. $Av_3 = \lambda_3 \cdot v_3$. OK.
Eigenvectors that correspond to different eigenvalues are linearly independent.

2. Consider the following subspace V of $\mathbb{R}^3$
$V = \text{SPAN}(v_1, v_2, v_3)$ where

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad v_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} \quad v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

(a) Show that $B = \{v_1, v_2\}$ is a basis of V.
v_1 and v_2 are clearly linearly independent. We need to show that $v_3 \in \text{SPAN}(\{v_1, v_2\})$:

$$v_3 = -v_1 - v_2$$

(b) For which real number x is the following vector w an element of V?

$$w = \begin{pmatrix} 2 \\ 3 \\ x \end{pmatrix}$$

Write w as a linear combination of $\{v_1, v_2\}$.
Compute the reduced row echelon form of the matrix $(v_1 \;\; v_2 \mid w)$ to find that $x = -5$ and $w = 2v_1 + 5v_2$.

(c) Let

$$u_1 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix} \quad u_2 = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \quad u_3 = \begin{pmatrix} -1 \\ -1 \\ 2 \end{pmatrix}.$$

Show that u_1, u_2, u_3 are elements of V. Give the coordinates of u_1, u_2, u_3 with respect to the basis B.
Compute the reduced row echelon form of the matrix $(v_1 \;\; v_2 \mid u_1 \;\; u_2 \;\; u_3)$ to find that $u_1 = 2v_1 + v_2$, $u_2 = -v_1 + v_2$, $u_3 = -v_1 - v_2$, so

$$[u_1]_B = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad [u_2]_B = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \quad [u_3]_B = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$$

(d) Let $T : V \to V$ be a linear map defined by

$$T(v_1) = u_1, \quad T(v_2) = u_2.$$

Give the matrix $[T]_{BB}$ of the linear map T with respect to the basis B of V. Hint: $\dim(V) = 2$ so this must be a 2 by 2 matrix.

$$([T(v_1)]_B \;\; [T(v_2)]_B) = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}.$$

(e) Compute: the rank and the nullity of T. If T is invertible, then compute the matrix of the inverse map T^{-1} with respect to the basis B.
Compute the inverse of $[T]_{BB}$ by computing the reduced row echelon form of the matrix $([T]_{BB} \mid I)$. The result is

$$([T]_{BB})^{-1} = [T^{-1}]_{BB} = \begin{pmatrix} 1/3 & 1/3 \\ -1/3 & 2/3 \end{pmatrix}.$$

The matrix $[T]_{BB}$ is invertible so the rank is 2 and the nullity is 0.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map given by

$$T\begin{pmatrix} x \\ y \end{pmatrix} = A \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x + 2y \end{pmatrix}$$

where

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Let $B = \{e_1, e_2\}$ and $B' = \{u_1, u_2\}$ where

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad u_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

(a) Compute A^{-1}, $\det(A)$ and $\det(A^{-1})$.
$\begin{pmatrix} 2/3 & -1/3 \\ -1/3 & 2/3 \end{pmatrix}$, 3 and 1/3.

(b) Compute the eigenvalues of A.
$\det(\lambda I - A) = \lambda^2 - 4\lambda + 3$ so the eigenvalues are 1 and 3.

(c) Give the B to B' change-of-basis matrix and the B' to B change-of-basis matrix.
B' to B is $\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$.
B to B' is the inverse of that, which is $\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$.

(d) Compute the following

i. $[T]_{BB}$. This is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

ii. $[T]_{BB'}$. This is $\begin{pmatrix} 3 & 1 \\ 3 & -1 \end{pmatrix}$.

iii. $[T]_{B'B}$. This is $\begin{pmatrix} 3/2 & 3/2 \\ 1/2 & -1/2 \end{pmatrix}$.

iv. $[T]_{B'B'}$. This is $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$. This matrix is diagonal which implies that the elements of B' are eigenvectors of A.