Linear Algebra, Test 4, April 15, 2004.

1. Let $B = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{pmatrix}$, and let $C = \begin{pmatrix} 3 & 1 \\ 2 & 1 \\ 1 & 1 \end{pmatrix}$.

   (a) (10 points). Compute the change of basis matrix from $B$ to $C$.

   (b) (5 points). If $[w]_B = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ then compute $[w]_C$ without computing $w$ itself.
2. Let $V$ be a vector space of dimension 3. True or false (2 points each):

(a) A set of four vectors in $V$ can never be linearly independent.

(b) A set of four vectors in $V$ can never be a spanning set of $V$.

(c) A set of two vectors in $V$ can never be linearly independent.

(d) A set of two vectors in $V$ can never be a spanning set of $V$.

(e) Any three linearly independent vectors in $V$ will always form a basis of $V$.

(f) A set of vectors in $V$ can only be a spanning set of $V$ if it contains three linearly independent vectors.

(g) A change of basis matrix is always invertible.

3. Let $P_1 = \{ a + bt | a, b \in \mathbb{R} \}$ be the vector space of all polynomials in $t$ of degree at most 1. Let $T : P_1 \to P_1$ be the linear map given by differentiation $T = d/dt$.

(a) (10 points). Let $B = 1, t$ be a basis of $P_1$. Let $A = [T]_B$. Compute $A$.

(b) (10 points). Compute all eigenvectors of $A$.

(c) (2 points). Is $A$ diagonalizable?

(d) (2 points). Does there exist a basis $C$ of $P_1$ for which $[T]_C$ is diagonal?
4. Let $V$ be a vector space with basis $B = b_1, b_2$ where $b_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, b_2 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$.

(a) (2 points). $V$ is a \ldots-dimensional subspace of $\mathbf{R}^{\ldots}$ (put numbers on the dots).

(b) (2 points). Consider the linear map $T : V \to V$ given by $T \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ x \end{pmatrix}$.

Compute $T(b_1)$ and $T(b_2)$.

(c) (8 points). Compute the matrix $[T]_B$.

(d) (10 points). Compute the eigenvectors of matrix $[T]_B$.

(e) (5 points). Give a basis $C$ of $V$ for which $[T]_C$ is a diagonal matrix.
5. Let 

\[ A = \begin{pmatrix} 1 & 6 \\ 1 & 0 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}. \]

(a) (6 points). Show that \( v_1 \) and \( v_2 \) are eigenvectors of \( A \), and give the corresponding eigenvalues \( \lambda_1, \lambda_2 \).

(b) (2 points). Compute the vectors \( A^{14}v_1 \) and \( A^{14}v_2 \) without computing any matrix-matrix or matrix-vector products, using only the fact that \( v_1, v_2 \) are eigenvectors and the fact that you know their eigenvalues from the previous question. 

(Hint: Multiplying \( A \) times \( v_1 \) is the same as multiplying \( \lambda_1 \) times \( v_1 \). Hence, multiplying \( A \) times \( A \) times \( v_1 \) is the same as multiplying \( \lambda_1 \) times \( \lambda_1 \) times \( v_1 \). So \( A^{2}v_1 = A Av_1 = \lambda_1 \lambda_1 v_1 = \lambda_1^2 v_1 \). So then \( A^{14}v_1 \) is \( \ldots \).)

(c) (6 points). Write the vector \( e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) as a linear combination of \( v_1, v_2 \).

(d) (6 points). Use the previous two questions to compute \( A^{14}e_1 \).

(e) (2 bonus points, only do this exercise if you have time left). A petri dish contains bacteria that are either 0-day old or 1-day old. The situation is described by a vector \( \begin{pmatrix} x \\ y \end{pmatrix} \) where \( x \) is the number of 0-day old bacteria, and \( y \) is the number of 1-day old bacteria. After every day, each 0-day old bacteria becomes 1-day old and produces one new 0-day old bacteria, this is described by 

\[ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} 1 \\ 1 \end{pmatrix} \]

while a 1-day old bacteria produces six new 0-day old bacteria and then dies, this is described by 

\[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 6 \\ 0 \end{pmatrix}. \]

Notice that this is precisely the action of matrix \( A \). Suppose we start with one 0-day old bacteria and no 1-day old bacteria, then after 14 days, how many bacteria will there be?