Linear algebra, take home quiz.

Turn in on Feb 12.

February 5, 2004

1. True or false?

   (a) If A is row equivalent to B (this means: we can row-reduce A to B) then the following two systems have the same solutions:
   $$AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad BX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}.$$

   (b) If A is an n by n matrix (note that a matrix that has the same number of rows as columns is called a *square matrix*) then A is row equivalent to the identity matrix if and only if
   $$AX = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$
   has only one solution (the trivial solution (the zero solution)). Note: the identity matrix is the matrix with 1's on the diagonal and 0's elsewhere. It is the only n by n matrix that is in reduced row echelon form and that has a pivot in every column.

2. Let
   $$u = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 2 \end{pmatrix}, \quad v = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 2 \end{pmatrix}, \quad \text{and} \quad b = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix},$$
   Give all (infinitely many) b_1, b_2, b_3, b_4 for which $b \in \mathrm{SPAN}\{u, v\}$.

3. Let
   $$A = \begin{pmatrix} 1 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \\ 4 & 6 & 7 \end{pmatrix}$$
   Compute the following:

   (a) The reduced row echelon form of A.

   (b) The rank of A.

   (c) Are the columns of A linearly dependent? If so, then give a linear relation between the columns.

   (d) Write down a basis for the SPAN of the columns of A. This means: write down a linearly independent set of vectors whose SPAN is the same as the SPAN of the columns of A.

4. Let

$$u_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad u_6 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

   (a) Compute the reduced row echelon form of $B = (u_1\ u_2\ u_3\ u_4\ u_5\ u_6)$.

   (b) Compute all solutions of the system $Bx = 0$.

   (c) Give a basis for each of the following vector spaces:
   $\mathrm{SPAN}(\{u_1\})$, $\mathrm{SPAN}(\{u_1, u_2\})$, $\mathrm{SPAN}(\{u_1, u_2, u_3\})$, $\mathrm{SPAN}(\{u_1, u_2, u_3, u_4\})$,
   $\mathrm{SPAN}(\{u_1, u_2, u_3, u_4, u_5\})$, $\mathrm{SPAN}(\{u_1, u_2, u_3, u_4, u_5, u_6\})$.

   (d) Whenever $\mathrm{SPAN}(\{u_1, u_2, \ldots, u_n\}) = \mathrm{SPAN}(\{u_1, u_2, \ldots, u_n, u_{n+1}\})$ express u_{n+1} as a linear combination of $u_1, u_2, \ldots, u_n$. So whenever a vector is in the SPAN of the previous vectors then write that vector as a linear combination of the previous vectors.

5. Let A be a 5 by 7 matrix for which the rank is 4. Compute the following:

   (a) The number of rows in $\mathrm{rref}(A)$ that are entirely zero.

   (b) The number of free variables in the system $Ax = 0$.

   (c) The number of basic variables.

   (d) The dimension of the SPAN of the columns of A (remember that the dimension is the number of elements of a basis).

   (e) Are the columns of A linearly dependent or independent?

   (f) Is the SPAN of the columns of A equal to $\mathbb{R}^5$?

   (g) Does there exist a right-hand side b for which $Ax = b$ is not consistent?

6. Write the function $\sin(x + \frac{\pi}{6})$ as a linear combination of $\cos(x), \sin(x)$. You may look up your trigonometry formulas.

7. Write the function $f(x) = x$ as a linear combination of the functions $g(x) = (x+1)^2$ and $h(x) = (x-1)^2$.

This quiz counts as part of your quiz/homework grade. The lowest test grade (but not the grade of the final!) can be replaced by your quiz/homework grade.

When you do this quiz, you may look in your notes and in the book. You may even ask other students for an explanation, but you should not simply copy their answers because if you get all the right answers, and on the test you can suddenly no longer do these same questions, then I will not count the result of this quiz. So the point is this: You may ask for help, you can even ask me for help (I can do an example in class that is similar the question you need help with), but you have to make sure you understand this material and can do the questions on your own before you turn this in next Thursday.