MAS 3301 Modern Algebra Homework Set 1

1. Prove that $\sqrt{3}$ is irrational. You will use the property that if a^2 is a multiple of 3, then a itself must be a multiple of 3. Explain why this is so.

2. Prove that $\sqrt{5}$ is irrational.

The Field Axioms:

Let $\mathbf{F}$ be a set with two binary operations $+$ and $\times$ that satisfy:

FA0: (closure) For all a, b in $\mathbf{F}$, $a + b$ and $a \times b$ are in $\mathbf{F}$.

FA1: (commutative) For all a, b in $\mathbf{F}$, $a + b = b + a$ and $a \times b = b \times a$.

FA2: (associative) For all a, b, c in $\mathbf{F}$, $a + (b + c) = (a + b) + c$ and $a \times (b \times c) = (a \times b) \times c$.

FA3: (identities) An additive identity 0 and a multiplicative identity 1 are in $\mathbf{F}$: for all a in $\mathbf{F}$, $a + 0 = a$ and $a \times 1 = a$.

FA4: (inverses) For all a and nonzero b in $\mathbf{F}$, there exists in $\mathbf{F}$ an additive inverse u for a and a multiplicative inverse v for b: $a + u = 0$ and $b \times v = 1$. Notation: u is denoted $-a$ and v is denoted b^{-1} or $\frac{1}{b}$.

FA5: (distributive) For all a, b, c in $\mathbf{F}$, $a \times (b + c) = a \times b + a \times c$.

3. Which of the field axioms do the natural numbers $\mathbf{N}$, the integers $\mathbf{Z}$, and the rational numbers $\mathbf{Q}$ satisfy?

4. Let $\mathbf{F} = \{x + y\sqrt{5} \colon x, y \in \mathbf{Q}\}$ with the usual addition and multiplication operations. Since $\mathbf{F}$ is a subset of the real numbers $\mathbf{R}$, FA1, FA2, and FA5 are automatically satisfied. Show that the remaining field axioms are satisfied so that $\mathbf{F}$ is a field.

6. For complex numbers a and b, prove that

$$\left| \frac{a - b}{1 - \bar{a}b} \right| = 1$$

if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?