MAS 3301 Modern Algebra Homework Set 3

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Definitions. A set $\mathbf{F}$ is a *ring* if addition and multiplication operations are defined on $\mathbf{F}$ that satisfy all the field axioms except that multiplicative inverses are not required to exist for every element of $\mathbf{F}$. The standard example is the ring of integers $\mathbf{Z}$. A nonzero element a of $\mathbf{F}$ is a *zero divisor* if there is a nonzero element b for which $ab = 0$. The ring of integers has no zero divisors. Exercise 2 below gives an example of a ring that does have zero divisors.

1. Let $\mathbf{F}$ be a ring.

   (i) Show that if a is a zero divisor of $\mathbf{F}$, then a fails to have a multiplicative inverse.

   (ii) Show that a field has no zero divisors.

2. The split-complex numbers are exactly like the complex numbers except that $i^2 = 1$. These numbers satisfy all the field axioms except that multiplicative inverses sometimes fail to exist. The multiplicative identity for the split-complex numbers is, as for the complex numbers, $1 = 1 + 0i$. The split-complex numbers do form a ring but not a field. It is helpful to denote split-complex numbers using the notation i_1 instead of notation i, to prevent mixing them up with the usual complex numbers.

   (i) Find the multiplicative inverse of $1 + 2i_1$ (which does exist), and show that $1 + i_1$ does not have a multiplicative inverse (Hint: for the second assertion use exercise 1(i)).

   (ii) Characterize those split-complex numbers $a + bi_1$ that do have a multiplicative inverse, ie, find conditions on a and b that determine whether or not $(a + bi_1)^{-1}$ exists.

   (iii) Find a general formula for $(a + bi_1)^{-1}$ when it exists.

3. The dual numbers are defined exactly like the complex and split-complex numbers except that $i^2 = 0$. Again, $1 = 1 + 0i$ is the multiplicative identity. We will denote dual numbers using the notation ϵ instead of notation i. This helps to prevent mixing them up with the complex numbers.

   (i) Find the multiplicative inverses of $1 + 2\epsilon$ and $1 + \epsilon$ (both exist). Show that ϵ^{-1} fails to exist.

   (ii) Characterize those dual numbers $a + b\epsilon$ that do have a multiplicative inverse, ie, find conditions on a and b that determine whether or not $(a + b\epsilon)^{-1}$ exists.

   (iii) Find a general formula for $(a + b\epsilon)^{-1}$ when it exists.

4. Let $\mathbf{D} = \{(a, b) \colon a, b \in \mathbf{R}\}$ with $(a, b) + (c, d) = (a+c, b+d)$ and $(a, b) \times (c, d) = (ac^2, bd^2)$.

   (i) Is $\times$ commutative? associative?

   (ii) The element I of $\mathbf{D}$ is a *right identity* if $z \times I = z$ for all z in $\mathbf{D}$ and a *left identity* if $I \times z = z$ for every z in $\mathbf{D}$. Find all the right identities in $\mathbf{D}$ (there are several).

   (iii) Are there any left identities in $\mathbf{D}$? Why or why not?