MAS 3301 Modern Algebra Homework Set 6

1. Identify the following elements of $\mathbf{F}_4$ as one of $0, 1, a, b$.

   (i) -1 (ii) $(1+a)^{-1}$ (iii) $\sqrt{b}$ (iv) $\sqrt[4]{a}$ (v) $\dfrac{a-1}{\sqrt[3]{b}}$

2. In any field $\mathbf{F}$, given a positive integer n and a field element $a \in \mathbf{F}$, let na denote the sum of n a's, ie,

   $$na = a + a + \cdots + a \qquad n \quad \text{summands.}$$

   Also, let $0a = \mathbf{0}$, where 0 is the integer zero and $\mathbf{0}$ is the additive identity in the field $\mathbf{F}$. The *characteristic* of the field $\mathbf{F}$, denoted $\chi(\mathbf{F})$, is the smallest value of $n \in \mathbf{N}$ such that $n\mathbf{1} = \mathbf{1} + \mathbf{1} + \cdots + \mathbf{1} = \mathbf{0}$, where we are using the symbol $\mathbf{1}$ to denote the multiplicative identity in $\mathbf{F}$. If no such n exists, we say that $\mathbf{F}$ has *characteristic zero*. What is the characteristic of $\mathbf{Z}/p$ if p is prime? What is the characteristic of $\mathbf{F}_4$? What is the characteristic of $\mathbf{Q}$? of $\mathbf{R}$? of $\mathbf{C}$?

3. Let $\mathbf{F}$ be a field of characteristic p. Explain why $pa = \mathbf{0}$ for every field element a in $\mathbf{F}$.

4. Let $\mathbf{F}$ be any field.

   (i) Let p be a positive integer and suppose there is a nonzero field element a such that $pa = \mathbf{0}$. Show that $p\mathbf{1} = \mathbf{0}$.

   (ii) Prove that if $p = \chi(\mathbf{F})$, then p is prime.

5. Use the Euclidean Algorithm to find the gcd of the following pairs of integers.

   (i) $91, 110$ (ii) $3630, 3822$ (iii) $3366, 7150$ (iv) $1950, 22638$

   (i) Ans=1 (ii) Ans=6 (iii) Ans=22 (iv) Ans=6

6. For each pair in Problem 5, write the gcd as an integer combination of the pair, ie, in the form $\gcd(a, b) = ma + nb$.

   (i) Ans: $m = -29, n = 24$ (iv) Ans: $m = -1010, n = 87$

7. Find the relatively prime pairs among the following.

   (i) $252, 539$ (ii) $360, 539$ (iii) $360, 1309$ (iv) $520, 539$ (v) $520, 693$

8. Find $\overline{252}^{-1}$ in $\mathbf{Z}/539$, if it exists.

9. Find $\overline{360}^{-1}$ in $\mathbf{Z}/539$, if it exists.

10. Find $\overline{360}^{-1}$ in $\mathbf{Z}/1309$, if it exists.