GRV II final.

1. Let R be an integral domain and let $r \in R$ be not zero and not a unit.
   (a) Give definition of: p is prime.
   (b) Give definition of: p is irreducible.
   (c) Which one (prime or irreducible) implies the other one?
2. Let $f \in \mathbb{Q}[x]$ be monic and not constant. Suppose that $e^2 = e$ has only two solutions in the ring $\mathbb{Q}[x]/(f)$. Show that $f = g^d$ for some $d \geq 1$ and some irreducible $g \in \mathbb{Q}[x]$.
3. Let p be a prime number.
   (a) Up to isomorphism, how many $\mathbb{Z}$-modules exist with precisely p^4 elements? List all.
   (b) Up to isomorphism, how many $\mathbb{F}_p[x]$-modules exist with precisely p^4 elements?
4. Let K be the splitting field of $x^6 - 2$ over $\mathbb{Q}$.
   (a) What is $[K : \mathbb{Q}]$? Explain.
   (b) How many subfields E does K have with $[E : \mathbb{Q}] = 4$?
5. Let p be a **prime number** > 2 and let $K = \mathbb{Q}(\zeta_p)$.
   (a) Show that K has precisely one subfield F with $[K : F] = 2$.
   (b) Show that K has precisely one subfield E with $[E : \mathbb{Q}] = 2$.
   (c) Show that $E \subset \mathbb{R}$ if and only if $p \equiv 1 \bmod 4$.
6. Let $K/\mathbb{Q}$ be Galois with group G and let $b \in K$ with $b \neq 0$. Show that there exists $\sigma \in G$ with $\sigma(b) = -b$ if and only if $b \notin \mathbb{Q}(b^2)$.