GRV II, test 1.

1. Let R be a commutative ring with identity. Write down definitions for: an irreducible element of R, a prime element of R, and give the definition of an Eisenstein polynomial $f \in R[x]$.

2. Let R be a commutative ring with identity.

   (a) Let K be a field and let $\phi : R \to K$ be a homomorphism with $\phi(1) \neq 0$. Show that the kernel of ϕ is a prime ideal.

   (b) Conversely, if P is a prime ideal, then show that there exists a field K and a homomorphism $\phi : R \to K$ with kernel P.

3. Let $n = p_1^{e_1} p_2^{e_2} p_3^{e_3}$ where p_1, p_2, p_3 are distinct prime numbers and $e_i > 0$. Show there are 8 distinct $m \in \{0, \ldots, n-1\}$ for which $m^2 \equiv m \bmod n$.

4. Suppose $f \in \mathbb{Z}[i][x]$ is reducible in the larger ring $\mathbb{Q}[i][x]$.
   Must f then also be reducible in the smaller ring $\mathbb{Z}[i][x]$?

5. List every (up to isomorphism) abelian group of order 128 that has a subgroup isomorphic to $C_2 \times C_2 \times C_2$ but not a subgroup isomorphic to $C_2 \times C_2 \times C_2 \times C_2$.

6. If p is a prime number and $p \equiv 1 \bmod 3$, then show that there exists a non-abelian group of order $3p$.

7. (Take home). Let R be a commutative ring with identity.
   Let I, J be ideals and let M be the R-module $R/I \times R/J$. Show that

$$I + J = R \iff M \text{ is a cyclic } R\text{−module.}$$