GRV II test 2.

1. (a) Let $R = \mathbb{Q}[x]$. List, up to isomorphism, every R-module M with the following properties:
   (i) The dimension of M as a $\mathbb{Q}$-vector space is 4.
   (ii) For $x^4 m = 0$ for every $m \in M$.

   (b) For each M in (i), give the Jordan Normal Form of the matrix of the $\mathbb{Q}$-linear map $M \to M$ given by $m \mapsto x \cdot m$.

2. Let M be a $\mathbb{Q}[x]$-module that is not cyclic. Suppose that the dimension of M as a $\mathbb{Q}$-vector space is 5. Show that there is an eigenvalue in $\mathbb{Q}$, i.e., show that there exists $\lambda \in \mathbb{Q}$ and $m \in M - \{0\}$ with $x \cdot m = \lambda m$.

   Give an example that shows this need not be true in dimension 4.

3. The map $\phi : \mathbb{Z}^3 \to \mathbb{Z}^3$ is given by the matrix

$$A := \begin{pmatrix} 42 & 24 & 36 \\ 24 & 14 & 20 \\ 36 & 20 & 32 \end{pmatrix}$$

   Compute the standard form (rank and invariant factors) of the $\mathbb{Z}$-module $\mathbb{Z}^3/\mathrm{im}(\phi)$.

4. Let R be a commutative ring, let $A \in \mathrm{Mat}_{n,n}(R)$, and $d = \det(A)$.

   (a) Suppose there is a non-zero $v \in R^n$ with $Av = 0$. Show that d is zero or a zero-divisor. (hint: adjoint matrix).

   (b) Let $w \in R^n$. Show that there exists $v \in R^n$ with $Av = dw$.

5. Let R be an integral domain. Let M be an R-module of rank $m < \infty$, and let $N \subseteq M$ be a submodule of rank n.

   (a) Give the definition of rank. M has rank m means:

   (b) Show that $n \leq m$.

   (c) Suppose that $m = n$. Prove that M/N is a torsion module (i.e. prove that if $x \in M/N$ then there exists $r \in R$, $r \neq 0$, and $rx = 0$).