GRV II test 2.

1. (a) Let $R = \mathbb{Q}[x]$. List, up to isomorphism, every R-module M with the following properties:
   (i) The dimension of M as a $\mathbb{Q}$-vector space is 4.
   (ii) For $x^4 m = 0$ for every $m \in M$.

   (b) For each M in (i), give the Jordan Normal Form of the matrix of the $\mathbb{Q}$-linear map $M \to M$ given by $m \mapsto x \cdot m$.

   M is isomorphic to $R/(a_1) \oplus \cdots \oplus R/(a_k)$ with $a_1 | a_2 | \cdots | a_k$. The dimension is 4 so the degree of $a_1 \cdots a_k$ is 4. Since x^4 annihilates M, we find that each a_i divides x^4. So the a_i are powers of x, and the sum of those powers is 4. This means that each $a_1, \ldots, a_k$ corresponds precisely to a partition of 4. The partitions of 4 are: $1+1+1+1$, $1+1+2$, $1+3$, $2+2$, and 4. The corresponding $a_1, \ldots, a_k$ are: (i) x, x, x, x (ii) x, x, x^2, (iii) x, x^3, (iv) x^2, x^2, and (v) x^4.

   The corresponding matrices in Jordan form are

$$\begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}$$

$$\begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0\\0&0&1&0\\0&0&0&1\\0&0&0&0 \end{pmatrix}$$

2. Let M be a $\mathbb{Q}[x]$-module that is not cyclic. Suppose that the dimension of M as a $\mathbb{Q}$-vector space is 5. Show that there is an eigenvalue in $\mathbb{Q}$, i.e., show that there exists $\lambda \in \mathbb{Q}$ and $m \in M - \{0\}$ with $x \cdot m = \lambda m$.

   Give an example that shows this need not be true in dimension 4.

   M is isomorphic to $R/(a_1) \oplus \cdots \oplus R/(a_k)$ with $a_1 | a_2 | \cdots | a_k$. Here $k > 1$ since M is not cyclic. If a_1 has degree 1, then it has a rational root $\lambda \in \mathbb{Q}$. The product $a_1 \cdots a_k$ is the characteristic polynomial, its roots are the eigenvalues, so any root of any a_i is an eigenvalue. So λ is an eigenvalue of the multiply-by-x map.
   If a_1 has degree 2, then k can only be 2 and a_2 must have degree 3. But a_2 must also be divisible by a_1, and the quotient a_2/a_1 is a polynomial of degree 1, which thus has a rational root.

   If the dimension is 4, then we could have $a_1 = a_2$ irreducible of degree 2. For instance, $a_1 = a_2 = x^2 + 1$. The matrix for this is is a block matrix, where each block on the diagonal is the companion matrix of x^2+1:

$$\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

3. The map $\phi : \mathbb{Z}^3 \to \mathbb{Z}^3$ is given by the matrix

$$A := \begin{pmatrix} 42 & 24 & 36 \\ 24 & 14 & 20 \\ 36 & 20 & 32 \end{pmatrix}$$

Compute the standard form (rank and invariant factors) of the $\mathbb{Z}$-module $\mathbb{Z}^3/\mathrm{im}(\phi)$.

The rank is 1 and the invariant factors $a_1, \dots, a_k$ are $2, 6$. So the module is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}/(2) \oplus \mathbb{Z}/(6)$.

4. Let R be a commutative ring, let $A \in \mathrm{Mat}_{n,n}(R)$, and $d = \det(A)$.

   (a) Suppose there is a non-zero $v \in R^n$ with $Av = 0$. Show that d is zero or a zero-divisor. (hint: adjoint matrix).

   (b) Let $w \in R^n$. Show that there exists $v \in R^n$ with $Av = dw$.

Let B be the adjoint matrix. Then $AB = BA = dI$. For the first question, multiply $Av = 0$ by B gives $dv = 0$. But v contains non-zero entries. Then d must be zero or a zero-divisor. For the second question, take $v = Bw$. Then $Av = ABw = dw$.

5. Let R be an integral domain. Let M be an R-module of rank $m < \infty$, and let $N \subseteq M$ be a submodule of rank n.

   (a) Give the definition of rank. M has rank m means:

   (b) Show that $n \leq m$.

   (c) Suppose that $m = n$. Prove that M/N is a torsion module (i.e. prove that if $x \in M/N$ then there exists $r \in R$, $r \neq 0$, and $rx = 0$).

The rank of M is the highest number of R-linearly independent elements of M. Since $N \subseteq M$, if N has n independent elements, then M will also have the same n independent elements. So $m \geq n$.
If $m = n$ then we have n independent elements $a_1, \dots, a_n$ in N, but we can not find additional independent elements in M. So if $x \in M$ then $a_1, \dots, a_n, x$ must be dependent: $c_1 a_1 + \cdots + c_n a_n + rx = 0$ where not all coefficients are 0. Then $r \neq 0$ because $a_1, \dots, a_n$ are independent. So $rx \in N$. Then rx is zero in M/N, for any $x \in M$, so M/N is torsion.