GRV II test 2a.

1. The map $\phi : \mathbb{Z}^3 \to \mathbb{Z}^3$ is given by the matrix

$$A := \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

   (a) Compute the standard form of the $\mathbb{Z}$-module $\mathbb{Z}^3/\mathrm{im}(\phi)$.
      - Rank:
      - Invariant factor(s):

   (b) What is the minimal number of generators for $\mathbb{Z}^3/\mathrm{im}(\phi)$?

2. Up to similarity, how many 4 by 4 matrices over $\mathbb{F}_p$ exist whose characteristic polynomial is not equal to its minimal polynomial?

3. Let A be an n by n matrix over $\mathbb{Q}$. If all eigenvalues are in $\mathbb{Q}$ then show that there exists a basis $b_1, \ldots, b_n$ of $\mathbb{Q}^n$ for which $Ab_1 \in \mathrm{SPAN}_{\mathbb{Q}}(b_1)$ and $Ab_i \in \mathrm{SPAN}_{\mathbb{Q}}(b_i, b_{i-1})$ for $i = 2, \ldots, n$.

4. Let m be a positive integer and $f = x^m - 2$. Show that f is irreducible. If A is an n by n matrix over $\mathbb{Q}$ and $A^m = 2I$ then show that $m|n$.

5. (bonus or take-home) (if exercises 1–4 are correct then you aced the test).

   Let R be a PID and let M be a finitely generated R-module with annihilator $(f) \neq (0)$. Show that there exists a homomorphism $\phi : M \to M$ with $\phi \circ \phi = \phi$ and $\phi(M) \cong R/(f)$.