GRV II test 3.

1. Let $f(x)$ be irreducible in $\mathbb{Q}[x]$ and $g(x)$ be irreducible in $K[x]$ where $[K:\mathbb{Q}]=d$. Suppose that $g|f$ (although f is irreducible in $\mathbb{Q}[x]$, it might factor in a larger ring $K[x]$).
   (a) Show that $\deg(f)$ divides $d\cdot\deg(g)$.
   Hint: extend K with a root of g.
   (b) Take-home: If K is Galois over $\mathbb{Q}$ then show that all irreducible factors of f in $K[x]$ have the same degree.
2. Let $k=\mathbb{Q}(\zeta_n)$ and let $a\in k$ and $K=k(a^{1/n})$.
   (a) Show that K/k is a Galois extension.
   (b) Take-home: Show that $\mathrm{Gal}(K/k)$ is a subgroup of C_n.
3. Let K be the splitting field of x^4-2 over $\mathbb{Q}$.
   Hint for (a)+(b): you can count them without computing them.
   (a) How many subfields $E\subset K$ have $[E:\mathbb{Q}]=4$?
   (b) How many of those subfields are Galois over $\mathbb{Q}$?
4. Suppose that $K\subset\mathbb{C}$ is finite extension of $\mathbb{Q}$ with degree $[K:\mathbb{Q}]=n$.
   (a) If $\sqrt[d]{2}\in K$ then show that $d|n$.
   (b) In the rest of this exercise, assume that $K/\mathbb{Q}$ is Galois with group G.
   If $\sqrt[d]{2}\in K$ then show that $\phi(d)|n$ where ϕ is the Euler ϕ function.
   (c) If G is abelian then show that $\sqrt[3]{2}\notin K$.
   (d) If G is cyclic then show that $\zeta_8\notin K$.
   (e) Take-home: show that $[K:K\bigcap\mathbb{R}]\le 2$.