Fields, take home questions.

1. Let $f \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 5 and let $g \in \mathbb{Q}[x]$ be an irreducible polynomial of degree 7. Let $\alpha \in \mathbb{C}$ be a root of f and $\beta \in \mathbb{C}$ be a root of g. Let $K_1 = \mathbb{Q}(\alpha)$, $K_2 = \mathbb{Q}(\beta)$ and $K_3 = \mathbb{Q}(\alpha, \beta)$.

   (a) Give: $[K_1 : \mathbb{Q}]$, $[K_2 : \mathbb{Q}]$, $[K_3 : \mathbb{Q}]$, $[K_3 : K_1]$ and $[K_3 : K_2]$.

   (b) Is f reducible or irreducible in $K_1[x]$? Why?

   (c) Is f reducible or irreducible in $K_2[x]$? Why?

2. Let p be a prime number, let $S = \{f(x) \in \mathbb{Q}[x] \mid f(x) \notin \mathbb{Q},\ \deg(f) < p\}$. Let $m(x)$ be any irreducible polynomial in $\mathbb{Q}[x]$ of degree p, and let $f(x)$ be any element of S. Show that there exists a unique $g(x) \in S$ for which $g(f(x)) - x$ is divisible by $m(x)$.
   Start as follows: Let α be a root of $m(x)$, let $\beta = f(\alpha)$, now prove that there exists a polynomial $g(x) \in S$ with $g(\beta) = \alpha$.

3. Let K be the splitting field of the polynomial $x^6 - 2$ over $\mathbb{Q}$. The Galois group G is isomorphic to $D_{2 \cdot 6}$ and can be written using two generators as follows $G =< \sigma, \tau >$, where σ is defined by $\sigma(\sqrt[6]{2}) = \zeta_6 \sqrt[6]{2}$, $\sigma(\zeta_6) = \zeta_6$, and τ is defined by $\tau(\sqrt[6]{2}) = \sqrt[6]{2}$, $\tau(\zeta_6) = \zeta_6^5$ (note: τ is complex conjugation). For each of the following subgroups H of G, write down the corresponding subfield K_H, the fixed field of H. You do not need to give proofs.

   (a) $H_1 = G$ (a group of order 12)

   (b) $H_2 = \{1\}$ (a group of order 1)

   (c) $H_3 =< \sigma >$ (a group of order 6)

   (d) $H_4 =< \tau >$ (a group of order 2)

   (e) $H_5 =< \sigma^2 >$ (a group of order 3)

   (f) $H_6 =< \sigma^2, \tau\sigma >$ (a group of order 6)

4. Let K and G be as in the previous question. What is the group $H \leq G$ belonging to the subfield $\mathbb{Q}(\sqrt{2})$?
   Hint: If $\sqrt{2}$ is an element of one of the fields you computed in the previous question, then the group H_i in that question will be a subgroup of the group H you need to find for this question. First check if this hint already gives you enough elements of H to generate H, if so, then write down those generators and you're done, if not, then you need to find more generators.